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On the Computational Complexity and Strategies of Online Ramsey Theory

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Abstract

An online Ramsey game $(\mathcal{G}, H)$ is a game between Builder and Painter, alternating in turns. In each round Builder draws an edge and Painter colors it either red or blue. Builder wins if after some round there is a monochromatic copy of the graph H, otherwise Painter is the winner. The rule for Builder is that after each his move the resulting graph must belong to the class of graphs $\mathcal{G}$. In this abstract we investigate the computational complexity of the related decision problem and we show that it is PSPACE-complete. Moreover, we study a generalization of online Ramsey game for hypergraphs and we provide a result showing that Builder wins the online Ramsey game for the case when $\mathcal{G}$ and H are 3-uniform hyperforests and H is 1-degenerate.

Keywords: online Ramsey game, complexity, strategy

1 Introduction

The online Ramsey game was introduced independently by Beck [1] and Friedgut et al. [4]. In this abstract we consider an extensions of the online Ramsey game $(\mathcal{G}, H)$ introduced by Grytczuk et al. [5] as follows. We are given a class of finite graphs $\mathcal{G}$ and a fixed graph H. There are two players, Builder and Painter, and the board of the game is an infinite independent set of vertices. In each round Builder draws a new edge and Painter colors it red or blue. The goal of Builder is to force Painter to create a monochromatic copy of H. However, in each round the graph induced by the edges must belong to $\mathcal{G}$, otherwise Builder loses. If Builder can always win the online Ramsey game $(\mathcal{G}, H)$, we say that H is *unavoidable* on $\mathcal{G}$. A graph class $\mathcal{D}$ is *unavoidable* on $\mathcal{C}$ if every graph $D \in \mathcal{D}$ is unavoidable on $\mathcal{C}$. The current graph during the gameplay is also called *background graph* and the graph H is called *goal graph*.

Let $R(H)$ be a *Ramsey graph* of H, that means a graph such that any two-coloring of its edges contains a monochromatic copy of H. If $R(H) \in \mathcal{G}$ then H is trivially unavoidable on $\mathcal{G}$. However, the question of unavoidability is usually highly nontrivial when $R(H) \notin \mathcal{G}$. For example, for $H = K_4$ and $\mathcal{G}$ the class of all 4-colorable graphs, no Ramsey graph $R(K_4)$ belongs to $\mathcal{G}$, but still K_4 is unavoidable on $\mathcal{G}$ by [5]. The online Ramsey theory is thus an interesting member of the family of Ramsey-related problems.

Online Ramsey theory recently gained popularity and different versions of the online Ramsey game were considered and several unavoidability results were obtained. Grytczuk et al. [5] has shown that forests are unavoidable on the class of all forests and that any k-colorable H is unavoidable on the class of k-colorable graphs. Conlon [3] studied the differences between standard Ramsey theory and the online theory. He proved that there is a constant c such that for infinitely many values k Builder needs to draw less then $c^{-k}\binom{r(k)}{2}$ edges to force K_k on the infinite complete background graph, where $r(k)$ is the standard Ramsey number for complete graphs. Petříčková [7] showed that the class of outerplanar graphs is unavoidable on the class of planar graphs. Butterfield et al. [2] presented several unavoidability results for graphs on the class of bounded-degree graphs. Rolnick [8] provided a complete characterization

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of trees unavoidable on the class of graphs with maximum degree 4.

In the standard Ramsey theory setting, the following natural question appeared: Given graphs F, G, H, what is the computational complexity of deciding whether for any red-blue coloring of edges of F there is a red copy of G or blue copy of H in F? This problem was intensively studied by Schaefer [9] who obtained several results. The most remarkable result of Schaefer is that this standard Ramsey decision problem is Π_2-complete.

Therefore, as online Ramsey theory is an important member of the family of Ramsey-related problems, it is a natural question to investigate the computational complexity of the decision version of the online Ramsey game problem: Given a partially colored graph G and a graph H, does Builder win the online Ramsey game (G, H)? We establish the computational complexity of this problem to be PSPACE-complete, as shown in Section 2.

Moreover, another natural task is to generalize online Ramsey theory results for various hypergraph classes. As far as we known, very little has been done in this field. Kierstead and Konjevod [6] studied Builder's strategies for complete hypergraphs. In Section 3 we provide a result showing that any 1-degenerate 3-uniform hyperforest H is unavoidable on the class of all 3-uniform hyperforests.

2 Complexity of the online Ramsey game

We first define the decision problem related to the online Ramsey game. Note that the background graph is induced on an infinite vertex set and the class $\mathcal{G}$ may be infinite as well. We have thus chosen the following way how to deal with this issue. The input of the decision problem in fact consists of two parts: the partially colored background graph and the goal graph, and the *rules*. Rules are defined as an algorithm (given as a binary encoding of a Turing machine, say) that is executed after every Builder's and Painter's move to check, whether they follow the rules and the background graph belongs to the class $\mathcal{G}$.

Problem:	ONLINE RAMSEY GAME
Input:	Partially edge-colored background graph G, goal graph H.
Rules:	Rules for Builder and Painter—rules have to be independent on the input and the decision problem if a move is valid belongs to PSPACE (measured against the size of the graph G).
Question:	Can Builder force a monochromatic copy of H in G?

It is easy to see that this problem is in PSPACE regardless of the rules. We may use a simple depth-first-search algorithm over the set of valid moves such that in each position the algorithm iterates over all valid moves and recursively searches each of the moves for a solution. The maximum depth of the recursion is linear in the size of G.

Theorem 2.1 ONLINE RAMSEY GAME *is PSPACE-complete.*

The fact that ONLINE RAMSEY GAME is PSPACE-complete corresponds with the complexity of many other decision problem regarding the existence of game strategies. However, ONLINE RAMSEY GAME is PSPACE-complete even for quite simple inputs as we will state later. For simplicity, we show the idea of the hardness proof of a slightly relaxed problem, which we call MULTIPLE ONLINE RAMSEY GAME. It is based on ONLINE RAMSEY GAME but the input also contains an integer parameter m and the question is changed to: Can Builder force m monochromatic copies of the goal graph H in the background graph G? Individual copies of H do not have to be painted by the same color.

The following two-player PAIRED FORMULA GAME played on propositional formulas will be reduced to ONLINE RAMSEY GAME. We are given a propositional formula φ in CNF with variables $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$, which are organized in pairs $(x_1, y_1), \ldots, (x_n, y_n)$. In each round the first player picks a previously unpicked pair (x_i, y_i) and sets x_i, then the seconds player sets y_i. The game ends when all variables are set. The first player wins if the formula φ is true at the end of the game, otherwise the second player wins.

Theorem 2.2 *The problem of deciding if the first player wins* PAIRED FORMULA GAME *is PSPACE-complete.*

In the reduction we will use the following graph. The (n, a, b)-*shackles* is a graph which contains two disjoint cycles C_a and C_b connected by a path P_n of n edges that does not cross C_a and C_b (with the exception of the endpoints). If n is even we call the center vertex of P_n the *center of shackles*. The edge of P_n adjacent to C_a (or C_b) is called *a-connection* (or *b-connection*). See Figure 1 for the illustration.

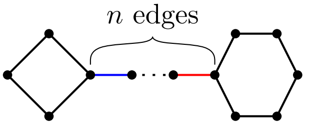

Fig. 1. Example of $(n, 4, 6)$-shackles with blue 4-connection and red 6-connection.

Theorem 2.3 MULTIPLE ONLINE RAMSEY GAME *is PSPACE-complete.*

Proof. [Idea of the proof] As we stated before, we reduce from PAIRED FORMULA GAME. Let a formula φ and pairs of φ-variables $(x_1, y_1), \ldots, (x_n, y_n)$ be the input for the formula game. The players of the formula game will be addressed as the first and the second player, and the players of the online Ramsey game will be addressed as Builder and Painter.

First, we describe the gadgets for variables. Let (x, y) be a variable pair. For all occurrences $x^1, \ldots, x^k, y^1, \ldots, y^\ell$ of variables x and y in φ we construct a corresponding $(4, 4, 6)$-shackles. We add one more vertex connected to the centers of all these shackles. See Figure 2 for an example, where it is also depicted how we precolored these gadgets.

The idea of the equivalence proof of the formula game and Ramsey game is as follows. If the first player sets variable x to 1 then in the corresponding gadget only 4-connections are drawn by Builder. If $x = 0$ then only 6-connections are drawn. If the second player sets the variable y to 1 then the drawn edges in the corresponding gadget are colored blue, and red for $y = 0$.

For consistency, we need to assure that in every gadget Builder picks edges only of one connection type and Painter colors all drawn edges by one color. We force it by two simple rules. Builder cannot pick an edge e such that the graph induced by all colored edges with the edge e contains $(4, 4, 6)$-shackles or $(6, 4, 6)$-shackles. Painter has to color the edges such that every $(6, 4, 4)$-shackles or $(6, 6, 6)$-shackles in a graph induced by the colored edges contains even number of red edges and even number of blue edges.

To finish the construction of the background graph G we add a red and blue 8-cycle connected by one vertex for every clause of the formula φ and connect them into the shackles from variable gadgets as depicted in Figure 2.

The goal graph H is defined as an 8-cycle with an attached path of 10 edges. We set the parameter m to be the number of clauses in φ. To finish the proof we need to show that the first player wins PAIRED FORMULA GAME if and only if Builder wins MULTIPLE ONLINE RAMSEY GAME. We can simply

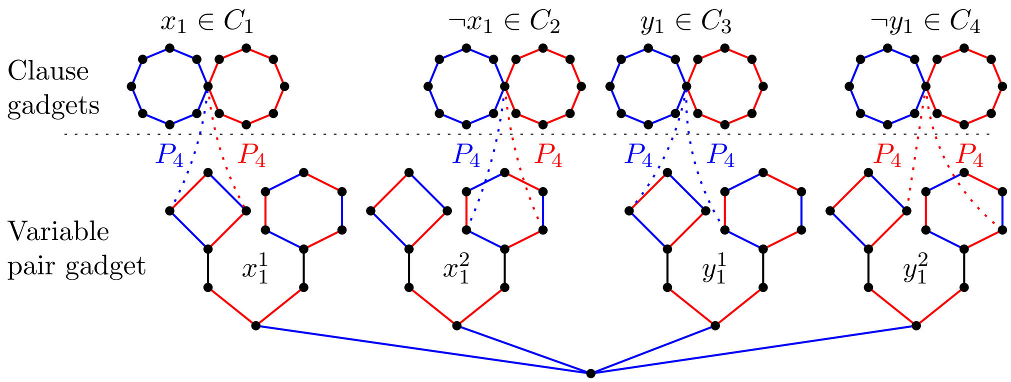

Fig. 2. Example of clause gadgets and the gadget for variable pair (x, y) when both variables (x, y) have two occurrences. Almost all edges are precolored, only 4- and 6-connections can be drawn by Builder. Every clause gadget is connected to the variable gadget by a path of length 4.

convert the winning strategy of the first player (or the second player) to the winning strategy of Builder (or Painter). When both games end, the clause C in φ is true if and only if there is a monochromatic copy of H in the colored G containing an 8-cycle from a gadget representing the clause C. □

Theorem 2.1 holds even for a bipartite background graph G with maximal degree 3, a tree as the goal graph H, and when Painter has no rule and valid Builder's moves are determined only by a finite set of finite forbidden graphs. Thus, valid Builder's moves can be checked in polynomial time. In some sense, this means Theorem 2.1 is optimal. If the background graph G has maximum degree 2 then G is a disjoint union of cycles and paths. Therefore, Builder can force a monochromatic path of length 2 if and only if he can build an odd cycle, which is easy to compute since the rules only forbid a finite set of graphs. Otherwise Builder can force only a monochromatic edge.

Rules for Builder are also important. If Builder has no rule restrictions, he may eventually draw all edges of the background graph. Therefore, Builder can win if and only if any edge 2-coloring of the background graph G contains a monochromatic copy of the goal graph H. However, this is the problem of classic Ramsey theory, which has been shown by Schaefer [9] to be Π_2-complete.

Even though Online Ramsey Game is PSPACE-complete in general, for a more restricted version of the game we may obtain NP-completeness, as shown in Theorem 2.5. By *t-star* we mean a complete bipartite graph $K_{1,t}$.

Problem:	STAR ONLINE RAMSEY GAME
Input:	Background graph G, $s, t \in \mathbb{N}$.
Rules:	Builder can build a vertex disjoint union of stars.
Question:	Can Builder force s copies of t-stars of the same color?

We obtain the following characterization of the Builder's win in STAR ONLINE RAMSEY GAME.

Lemma 2.4 *Builder wins* STAR ONLINE RAMSEY GAME *if and only if the background graph G contains $2s-1$ vertex disjoint $2t-1$-stars.*

The problem if a graph G contains s vertex disjoint t-stars is in NP and can be easily reduced to INDEPENDENT SET. Therefore, we obtain the following result.

Theorem 2.5 STAR ONLINE RAMSEY GAME *is NP-complete.*

3 Unavoidability of hypertrees

Grytczuk et al. [5] proved that forests are unavoidable on the class of all forests. The natural question is thus to generalize this online Ramsey theory result for hyperforests. Here we present a hypergraph class which is unavoidable on 3-uniform hyperforests.

Let $H = (V, F)$ be a hypergraph and G be a graph. Graph G is a *host graph* for H if every hyperedge $f \in F$ induces a connected subgraph in G. The hypergraph H is a *hyperforest* if there exists a forest which is the host graph of H.

Theorem 3.1 *The 1-degenerate 3-uniform hyperforests are unavoidable on 3-uniform hyperforests.*

The proof is done by induction over the edges. In the induction step we remove the edge with the vertex of degree 1 and apply the induction hypothesis. Also note that the goal hypergraph must be uniform in order to be unavoidable; otherwise Painter could paint edges of different size by different colors.

Further work

So far we have not been able to generalize Theorem 3.1 for general k-uniform hypertrees, which is thus our next effort. The online Ramsey theory for hypergraphs is a field with almost no results and we feel there is a lot of potential for future research. Also, the generalization of the online Ramsey theory for more colors than two seems to be surprisingly hard in many cases.

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