



# Counting in hypergraphs via regularity inheritance

Ewan Davies<sup>1</sup>

*Department of Mathematics  
London School of Economics and Political Science  
London WC2A 2AE, UK*

---

## Abstract

We develop a theory of regularity inheritance in 3-uniform hypergraphs. As a simple consequence we deduce a strengthening of a counting lemma of Frankl and Rödl. We believe that the approach is sufficiently flexible and general to permit extensions of our results in the direction of a hypergraph blow-up lemma.

*Keywords:* Hypergraph regularity, counting lemma, regularity inheritance

---

## 1 Introduction

Szemerédi's regularity lemma [15] states that any graph  $G$  has a 'regular partition' into a bounded number of pieces, almost all of which are quasirandom. An accompanying counting lemma tells us that the number of copies of any small subgraph in  $G$  is approximately the same as would be expected if the pieces were genuinely random. There are various generalisations to hypergraphs (e.g. [6,7,14]) which have been instrumental in solving important problems.

---

<sup>1</sup> Email: [E.S.Davies@lse.ac.uk](mailto:E.S.Davies@lse.ac.uk)

The blow-up lemma of Komlós, Sárközy and Szemerédi [9] is a powerful development of the counting lemma. It uses a randomised embedding algorithm to show that we can actually embed certain spanning subgraphs into suitable pieces of the partition produced by the regularity lemma. A technical condition called ‘image restrictions’ is permitted; a useful feature for applications, e.g. [1].

Keevash [8] proved a hypergraph blow-up lemma, but there are significant technical difficulties in establishing and applying the result. Complexity arises from the kind of hypergraph regularity used, known as the regular approximation lemma of Rödl and Schacht [13]. Furthermore, Keevash’s result only allows for a weak form of image restriction which makes it hard to use.

We describe a simple proof of a counting lemma for 3-uniform hypergraphs (3-graphs) using regularity inheritance, a technique we develop. The merits of our approach are connected with the simple kind of 3-graph regularity we use. A hypergraph blow-up lemma proved using these ideas is a work in progress. In particular, we seek to allow a number of vertices and pairs to be suitably image restricted, stronger than the restriction permitted by Keevash [8].

In future work we seek to generalise our techniques to  $k$ -uniform hypergraphs for  $k > 3$ . There are significant difficulties here, for some of the tools we use to establish regularity inheritance lemmas are not yet well-developed in higher uniformities. In particular a local characterisation of hypergraph regularity is important for our methods, and for algorithmic proofs. We also hope that our approach will generalise to the ‘sparse setting’, that is, when we work with a hypergraph which is a relatively dense subgraph of a sparse but pseudorandom hypergraph. A counting lemma for this setting was recently proved by Conlon, Fox and Zhao [3], but no counterpart blow-up lemma is known.

## 2 Results

Our notation resembles that of [3]. We identify graphs and 3-graphs with their edge sets. The neighbourhood of a vertex  $x$  in a graph or 3-graph  $F$  is  $F(x) := \{e \setminus x : x \in e \in F\}$ , and we write  $F[W]$  for the induced subgraph on vertex set  $W$ . Let  $\partial e = \{f \subsetneq e : |f| = |e| - 1\}$ . Functions  $g_f, h_e$  are the indicators for edge sets of graphs  $G$  and 3-graphs  $H$  respectively. All sets are finite. For an index set  $J$  and vertex sets  $\{V_j\}_{j \in J}$ , let  $V_J := \prod_{j \in J} V_j$ . We write  $x_J$  for an element of  $V_J$ , that is, the vector  $(x_j)_{j \in J}$  with  $x_j \in V_j$ . Write  $\mathbb{E}[g(x_J) | x_J \in V_J]$  for the expectation over  $x_J$  chosen uniformly at random from  $V_J$ . For statements involving positive real parameters, let  $a = b \pm c$  mean  $b - c \leq a \leq b + c$ , and write  $a \ll b$  to mean there is an increasing function  $f$  so that the argument is valid for  $0 < a < f(b)$ .

**Definition 2.1** Let  $f = \{1, 2\}$ , and  $\{V_j\}_{j \in f}$  be vertex sets. Let  $G$  be a bipartite graph with bipartition  $V_1, V_2$  and indicator function  $g_f: V_f \rightarrow \{0, 1\}$ . We say  $g_f$  is  $(\varepsilon_2, d_2)$ -regular if, for all  $j \in \partial f$  and functions  $u_j: V_j \rightarrow [0, 1]$ , we have

$$\left| \mathbb{E} \left[ \left( g_f(x_f) - d_2 \right) u_1(x_1) u_2(x_2) \mid x_f \in V_f \right] \right| \leq \varepsilon_2.$$

Although this definition looks stronger than the ‘usual’ definition, where each  $u_j$  takes values in  $\{0, 1\}$ , it is easy to prove that they are equivalent.

**Definition 2.2** Let  $e = \{1, 2, 3\}$  and  $\{V_j\}_{j \in e}$  be vertex sets. Let  $G$  be a tripartite graph with vertex partition  $V_1, V_2, V_3$ , and indicator functions  $g_f: V_f \rightarrow \{0, 1\}$  which are  $(\varepsilon_2, d_2)$ -regular for  $f \in \partial e$ . Let  $H$  be a 3-graph with indicator function  $h_e \leq \prod_{f \in \partial e} g_f$  pointwise, so edges of  $H$  are triangles in  $G$ . We say  $h_e$  is  $(\varepsilon_3, d_3)$ -regular relative to  $G$  if, for all  $f \in \partial e$  and functions  $u_f: V_f \rightarrow [0, 1]$  with  $u_f \leq g_f$  pointwise, we have

$$\left| \mathbb{E} \left[ \left( h_e(x_e) - d_3 \right) \prod_{f \in \partial e} u_f(x_f) \mid x_e \in V_e \right] \right| \leq \varepsilon_3 d_3^2.$$

We also say  $H$  is  $(\varepsilon_3, d_3)$ -regular with respect to  $G$  when we do not explicitly define the indicator function  $h_e$ .

Our main result is a strengthening of the 3-graph counting lemma developed previously by Rödl and coauthors Frankl, Nagle, Peng and Skokan [6, 11, 12]. These results require a stronger form of regularity than the above Definition 2.2 in which the edges of  $H$  are approximately uniformly distributed over triangles in unions of  $r$  subgraphs of  $G$ , for some  $r \in \mathbb{N}$  which is large compared to  $1/d_2$ . Taking  $r = 1$  in this formulation results in a strictly weaker regularity which is equivalent to our setting. This appeal of working with the weaker 3-graph regularity is a local characterisation in terms of subgraph counts. This characterisation does not hold for large values of  $r$  in the stronger 3-graph regularity, thus working with  $r = 1$  recovers the useful property; see [4, 10].

Our approach differs substantially from that of Nagle and Rödl [11] and we remove the need to apply any regularity lemma within the proof of our counting lemma. Another approach, due to Peng, Rödl and Skokan [12] is to transfer to the ‘dense’ or ‘absolute quasirandom’ setting in which  $d_2 = 1$  and appeal to a dense counting lemma. This idea is relevant to our work, as we depend on a technical result that is proved by a similar transference [10]. Since we work with the weaker regularity of Definition 2.2 our methods are stronger than those of [11, 12]. Moreover, our proof resembles an embedding process used in the blow-up lemma, a feature we hope to exploit in due course.

**Theorem 2.3 (Counting lemma)** *Let  $J$  be a set and  $F$  be a 3-graph on  $J$ . Write  $\partial F$  for the union of  $\partial e$  over  $e \in F$ . Let  $\{V_j\}_{j \in J}$  be vertex sets each of size at least  $n$ . For constants  $\frac{1}{n} \ll \varepsilon_2 \ll d_2 \ll \varepsilon_3 \ll \varepsilon'_3 \ll d_3$ , the following holds. Let  $G$  be a graph with indicators  $g_f: V_f \rightarrow \{0, 1\}$  which are  $(\varepsilon_2, d_2)$ -regular for all  $f \in \partial F$ . Let  $H$  be a 3-graph with indicators  $h_e: V_e \rightarrow \{0, 1\}$  which are  $(\varepsilon_3, d_3)$ -regular with respect to  $G$  for all  $e \in F$ .*

*Then*

$$\mathbb{E} \left[ \prod_{e \in F} h_e(x_e) \mid x \in V_J \right] = d_3^{|F|} d_2^{|\partial F|} \pm \varepsilon'_3 d_2^{|\partial F|}.$$

To obtain Theorem 2.3 we adapt the following sketch of the counting lemma in dense graphs. For simplicity, consider  $G$  from the setup of Theorem 2.3 and let  $\partial F$  be a triangle on vertices  $\{1, 2, 3\}$ . We apply a fact commonly known as ‘slicing’ or ‘regular restriction’ to  $G[V_2, V_3]$ , which is is  $(\varepsilon_2, d_2)$ -regular. If  $W_j \subset V_j$  for  $j = 2, 3$  are subsets of size at least  $\alpha|V_j|$ , the induced subgraph  $G[W_2, W_3]$  is  $(\varepsilon_2/\alpha^2, d_2)$ -regular. If the  $W_j$  are  $G$ -neighbourhoods of a typical  $x_1 \in V_1$ , we would have  $\alpha = d_2 \pm \varepsilon_2$  and so given  $\varepsilon_2 \ll d_2$ , we say the neighbourhood *inherits regularity*. To count triangles, sum over  $x_1 \in V_1$  the number of edges in  $G[G(x_1)]$ , which we estimate by the inherited regularity.

Applying the regularity lemma of Frankl and Rödl [5], one can only hope to achieve the relation  $d_2 \ll \varepsilon_3$  between parameters in Definition 2.2. Hence  $G$ -neighbourhoods are typically too small for the regularity of  $H$  to directly control edges upon them. An analogue of this problem also occurs in sparse graphs, however Conlon, Fox and Zhao [2] proved a form of regularity inheritance for that setting. We generalise their approach to hypergraphs and prove new inheritance lemmas such as 2.4 below. We deduce Theorem 2.3 from these lemmas by an appropriate generalisation of the above sketch for graphs.

To state an inheritance lemma, consider the setup of Theorem 2.3 and let  $0 \in J$ . We show that almost all  $x_0 \in V_0$  have the property that  $H[G(x_0)]$  is regular with respect to  $G[G(x_0)]$ . The parameter measuring the inherited regularity is  $\varepsilon'_3 \ll d_3$ , hence this result is much stronger than what one can deduce directly from Definition 2.2 using the  $u_f$  to indicate edges of  $G[G(x_0)]$ .

**Lemma 2.4 (3-sided inheritance)** *Let  $J = \{0, 1, 2, 3\}$ ,  $e = \{1, 2, 3\}$ , and  $F$  be the complete graph on  $J$ . Let  $\frac{1}{n} \ll \varepsilon_2 \ll d_2 \ll \varepsilon_3 \ll \varepsilon'_3 \ll d_3$ , and  $\{V_j\}_{j \in J}$  be vertex sets each of size at least  $n$ . Let  $G$  be a graph with indicators  $g_f: V_f \rightarrow \{0, 1\}$  which are  $(\varepsilon_2, d_2)$ -regular for  $f \in F$ , and let  $H$  be a 3-graph on  $V_e$  with indicator  $h_e \leq \prod_{f \in \partial e} g_f$  which is  $(\varepsilon_3, d_3)$ -regular with respect to  $G$ .*

*Then for all but at most  $\varepsilon'_3|V_0|$  vertices  $x_0 \in V_0$ , the induced 3-graph  $H[G(x_0)]$  is  $(\varepsilon'_3, d_3)$ -regular with respect to  $G[G(x_0)]$ .*

## References

- [1] Böttcher, J., M. Schacht and A. Taraz, *Proof of the bandwidth conjecture of Bollobás and Komlós*, Math. Ann. **343** (2009), 175–205.
- [2] Conlon, D., J. Fox and Y. Zhao, *Extremal results in sparse pseudorandom graphs*, Adv. Math. **256** (2014), 206–290.
- [3] Conlon, D., J. Fox and Y. Zhao, *A relative Szemerédi theorem*, Geom. Funct. Anal. (2015), 1–30.
- [4] Dementieva, Y., P. E. Haxell, B. Nagle and V. Rödl, *On characterizing hypergraph regularity*, Random Structures & Algorithms **21** (2002), 293–335.
- [5] Frankl, P. and V. Rödl, *The Uniformity Lemma for hypergraphs*, Graphs and Combinatorics **8** (1992), 309–312.
- [6] Frankl, P. and V. Rödl, *Extremal problems on set systems*, Random Structures & Algorithms **20** (2002), 131–164.
- [7] Gowers, W. T., *Quasirandomness, counting and regularity for 3-uniform hypergraphs*, Combin., Probab. Comput. **15** (2006), 143–184.
- [8] Keevash, P., *A hypergraph blow-up lemma*, Random Structures & Algorithms **39** (2011), 275–376.
- [9] Komlós, J., G. N. Sárközy and E. Szemerédi, *Blow-up lemma*, Combinatorica **17** (1997), 109–123.
- [10] Nagle, B., A. Poerschke, V. Rödl and M. Schacht, *Hypergraph regularity and quasi-randomness*, in: *Proc. of the 20<sup>th</sup> Annual ACM-SIAM Symp. on Discrete Algorithms*, 2009, 227–235.
- [11] Nagle, B. and V. Rödl, *Regularity properties for triple systems*, Random Structures & Algorithms **23** (2003), 264–332.
- [12] Peng, Y., V. Rödl and J. Skokan, *Counting small cliques in 3-uniform hypergraphs*, Combin., Probab. Comput. **14** (2005), 371–413.
- [13] Rödl, V. and M. Schacht, *Regular partitions of hypergraphs: regularity lemmas*, Combin., Probab. Comput. **16** (2007), 833–885.
- [14] Rödl, V. and J. Skokan, *Regularity Lemma for  $k$ -uniform hypergraphs*, Random Structures & Algorithms **25** (2004), 1–42.
- [15] Szemerédi, E., *Regular partitions of Graphs*, in: *Problème Combinatoire et Théorie des Graphes*, (Colloq. Internat. CNRS, Univ. Orsay, Orsay 1976) **260**, 1978, 399–401.