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The Diameter of Cyclic Kautz Digraphs

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Abstract

A prominent problem in Graph Theory is to find extremal graphs or digraphs with restrictions in their diameter, degree and number of vertices. Here we obtain a new family of digraphs with minimal diameter, that is, given the number of vertices and degree there is no other digraph with a smaller diameter. This new family is called modified cyclic digraphs $MCK(d,\ell)$ and it is derived from the Kautz digraphs $K(d,\ell)$.

It is well-known that the Kautz digraphs $K(d,\ell)$ have the smallest diameter among all digraphs with their number of vertices and degree. Here we define the cyclic Kautz digraphs $CK(d,\ell)$, whose vertices are labeled by all possible sequences $a_1 \dots a_\ell$ of length ℓ, such that each character a_i is chosen from an alphabet containing $d+1$ distinct symbols, where the consecutive characters in the sequence are different (as in Kautz digraphs), and now also requiring that $a_1 \neq a_\ell$. The cyclic Kautz digraphs $CK(d,\ell)$ have arcs between vertices $a_1 a_2 \dots a_\ell$ and $a_2 \dots a_\ell a_{\ell+1}$, with $a_1 \neq a_\ell$, $a_2 \neq a_{\ell+1}$, and $a_i \neq a_{i+1}$ for $i = 1, \dots, \ell-1$. The cyclic Kautz digraphs $CK(d,\ell)$ are subdigraphs of the Kautz digraphs $K(d,\ell)$.

We give the main parameters of $CK(d,\ell)$ (number of vertices, number of arcs, and diameter). Moreover, we construct the modified cyclic Kautz digraphs $MCK(d,\ell)$ to obtain the same diameter as in the Kautz digraphs, and we show that $MCK(d,\ell)$ are d-out-regular. Finally, we compute the number of vertices of the iterated line digraphs of $CK(d,\ell)$.

Keywords: Kautz digraphs, diameter, line digraphs, partial line digraphs.

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1 Introduction

A prominent problem in Graph Theory is to find extremal graphs or digraphs satisfying one or more restrictions in their diameter, degree and number of vertices. In this paper, we obtain a new family of digraphs with minimal diameter, in the sense that given the number of vertices and degree there is no other digraph with a smaller diameter. This new family is called modified cyclic Kautz digraphs $MCK(d,\ell)$ and it is derived from the family of the Kautz digraphs $K(d,\ell)$.

It is well-known that the Kautz digraphs $K(d,\ell)$, where d is the degree, have vertices labeled by all possible sequences $a_1 \dots a_\ell$ of length ℓ with different consecutive symbols, $a_i \neq a_{i+1}$ for $i = 1,\dots,\ell-1$, from an alphabet of $d+1$ distinct symbols. In this paper, we define the cyclic Kautz digraphs $CK(d,\ell)$ (see Figure 1), where the labels of their vertices are defined as the ones of the Kautz digraphs, with the additional requirement that the first and the last symbol must be different ($a_1 \neq a_\ell$). The cyclic Kautz digraphs $CK(d,\ell)$ have arcs between vertices $a_1a_2\dots a_\ell$ and $a_2\dots a_\ell a_{\ell+1}$, with $a_1 \neq a_\ell$, $a_2 \neq a_{\ell+1}$, and $a_i \neq a_{i+1}$ for $i = 1,\dots,\ell-1$. By this definition, we observe that the cyclic Kautz digraphs $CK(d,\ell)$ are subdigraphs of the Kautz digraphs $K(d,\ell)$. Unlike in Kautz digraphs $K(d,\ell)$, any label of a vertex of $CK(d,\ell)$ can be cyclically shifted to form again a label of a vertex of $CK(d,\ell)$. In contrast to the Kautz digraphs, the cyclic Kautz digraphs $CK(d,\ell)$ are not d-regular (neither d-out-regular). Therefore, for $CK(d,\ell)$ the meaning of d is the size of the alphabet minus one. Moreover, for $\ell > 3$, d also corresponds to the

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maximum out-degree of $CK(d, \ell)$.

The cyclic Kautz digraphs $CK(d, \ell)$ are related to cyclic codes. A linear code C of length ℓ is called cyclic if, for every codeword $c = (c_1, \dots, c_\ell)$, the codeword $(c_\ell, c_1, \dots, c_{\ell-1})$ is also in C. This cyclic permutation allows to identify codewords with polynomials. For more information about cyclic codes, see Van Lint [4] (Chapter 6).

The Kautz digraphs $K(d, \ell)$ can also be defined as iterated line digraphs of the complete symmetric digraphs K_{d+1} (see Fiol, Yebra and Alegre [2]). This also means that $K(d, \ell)$ can be obtained as the line digraph of $K(d, \ell - 1)$.

Fiol and Lladó defined in [1] the partial line digraph $PL(G)$ of a digraph G, where some (but not necessarily all) of the arcs in G become vertices in $PL(G)$. They proved that, if G is a d-out-regular digraph ($d > 1$) with order N and diameter D, the order N_{PL} and diameter D_{PL} of a partial line digraph $PL(G)$ satisfy $N \leq N_{PL} \leq dN$ and $D \leq D_{PL} \leq D+1$, respectively. Moreover, they showed that $PL(G)$ is also d-out-regular.

For a comparison between the partial digraph technique and other construction techniques to obtain digraphs with minimum diameter see Miller, Slamin, Ryan and Baskoro [6]. Since these techniques are related to the degree/diameter problem, we also refer to the comprehensive survey of this problem by Miller and Širáň [5].

This paper is organized as follows. We give in Section 2 the main parameters of the cyclic Kautz digraphs $CK(d, \ell)$, that is, the number of vertices, number of arcs, and diameter. Then, in Section 3, we construct the modified cyclic Kautz digraphs $MCK(d, \ell)$ in order to obtain digraphs with the same diameter as the Kautz digraphs, and we show that $MCK(d, \ell)$ are d-out-regular. Finally, in Section 4, we obtain the number of vertices of the t-iterated line digraph of $CK(d, \ell)$ for $1 \leq t \leq \ell - 2$, and for the case of $CK(d, 4)$ for all values of t. For the particular case of $CK(2, 4)$, these numbers of vertices follow a Fibonacci sequence. With the line digraph technique, we obtain digraphs with small diameter, small out-degrees and large number of vertices.

We use the habitual notation for digraphs, that is, a *digraph* $G = (V, E)$ consists of a (finite) set $V = V(G)$ of vertices and a set $E = E(G)$ of arcs (directed edges) between vertices of G. As the initial and final vertices of an arc are not necessarily different, the digraphs may have *loops* (arcs from a vertex to itself), but not *multiple arcs*, that is, there is at most one arc from each vertex to any other. If $a = (u, v)$ is an arc between vertices u and v, then vertex u (and arc a) is adjacent to vertex v, and vertex v (and arc a) is adjacent from v. Let $\Gamma^+_G(v)$ and $\Gamma^-_G(v)$ denote the set of vertices adjacent from and to

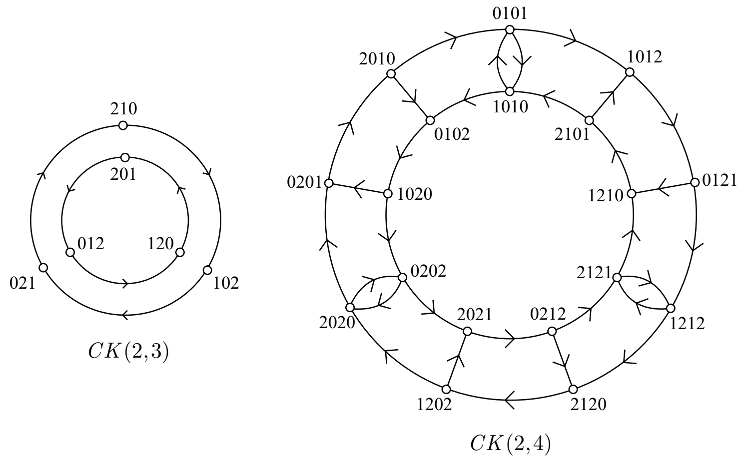


Fig. 1. The cyclic Kautz digraphs $CK(2,3)$ and $CK(2,4)$.

vertex v, respectively. Their cardinalities are the *out-degree* $\delta_G^+(v) = |\Gamma_G^+(v)|$ of vertex v, and the *in-degree* $\delta_G^-(v) = |\Gamma_G^-(v)|$ of vertex v. Digraph G is called *d-out-regular* if $\delta_G^+(v) = d$ for all $v \in V$, *d-in-regular* if $\delta_G^-(v) = d$ for all $v \in V$, and *d-regular* if $\delta_G^+(v) = \delta_G^-(v) = d$ for all $v \in V$.

We omit all the proofs because of space limitations.

2 Parameters of the cyclic Kautz digraphs

2.1 Number of vertices and number of arcs

Proposition 2.1 *The number of vertices of the cyclic Kautz digraph $CK(d,\ell)$ is $(-1)^\ell d + d^\ell$.*

Proposition 2.2 *The number of arcs of the cyclic Kautz digraph $CK(d,\ell)$ for $\ell \geq 3$ is*

$$(d+1)d^\ell - (2d-1)[(-1)^{\ell-1}d + d^{\ell-1}].$$

2.2 Diameter

First, let us fix $\{0, 1, \ldots, d\}$ to be the alphabet of $CK(d,\ell)$. We introduce the concept of the *disc representation* of a vertex of a cyclic Kautz digraph. The label of a vertex v of a cyclic Kautz digraph $CK(d,\ell)$ can be written circularly on a disc with a marked start. We refer to this as a *disc representation* of the vertex v (in short, the *disc of* v). Then, there is a path from a vertex u to a vertex v in $CK(d,\ell)$ if and only if the disc of v can be obtained from the disc of u by a sequence of the two following operations: Rotation of the disc, and swap of one symbol (in a valid way). In particular, in those cases

<table>
<tr><th>d \ ℓ</th><th>1</th><th>2</th><th>3</th><th>4</th><th>$\geqslant 5$</th></tr>
<tr><td>1</td><td rowspan="4">ℓ</td><td>1</td><td>$\nexists$</td><td>1</td><td>$\nexists$ / 1</td></tr>
<tr><td>2</td><td rowspan="3"></td><td>∞</td><td></td><td>∞</td></tr>
<tr><td>3</td><td colspan="3">2ℓ-1</td></tr>
<tr><td>$\geqslant 4$</td><td></td><td colspan="2">2ℓ-2</td></tr>
</table>

Fig. 2. Summary of the diameter of $CK(d, \ell)$, depending on the values of d and ℓ.

where $CK(d, \ell)$ is not connected (and thus the diameter is ∞), considering vertices in the disc representation helps us to identify 2 vertices of different components.

To obtain the diameter of $CK(d, \ell)$ we distinguish some cases, depending on the value of d and ℓ. For each case, we show that there is a path of a certain length between every pair of vertices, and we give two vertices at this distance. We summarize the diameter of $CK(d, \ell)$ in the following result (see Figure 2 for an overview).

Theorem 2.3 *The diameter of $CK(d, 1)$ is 1. The diameter of $CK(1, \ell \geq 2)$ is 1 if ℓ is even ($CK(1, \ell \geq 2)$ does not exist if ℓ is odd). The diameter of $CK(d \geq 2, 2)$ is 2. The diameter of $CK(2, 3)$ is infinite. The diameter of $CK(2, 4)$ is 7 ($= 2\ell - 1$). The diameter of $CK(2, \ell \geq 5)$ is infinite. The diameter of $CK(d \geq 3, 3)$ is 5 ($= 2\ell - 1$). The diameter of $CK(3, \ell \geq 3)$ is $2\ell - 1$. Finally, the diameter of $CK(d \geq 4, \ell \geq 4)$ is $2\ell - 2$.*

3 The modified cyclic Kautz digraphs

Recall that the diameter of the Kautz digraphs is optimal, that is, for a fixed out-degree d and number of vertices $(d+1)d^{\ell-1}$, the Kautz digraph $K(d, \ell)$ has the smallest diameter ($D = \ell$) among all digraphs with $(d+1)d^{\ell-1}$ vertices and degree d (see Li, Lu and Su [3]). Since the diameter of the cyclic Kautz digraphs $CK(d, \ell)$ is greater than the diameter of the Kautz digraphs $K(d, \ell)$, we construct the *modified cyclic Kautz digraphs* $MCK(d, \ell)$ by adding some arcs to $CK(d, \ell)$, in order to obtain the same diameter as $K(d, \ell)$.

In cyclic Kautz digraphs $CK(d, \ell)$, a vertex $a_2 \ldots a_{\ell+1}$ is forbidden if $a_2 = a_{\ell+1}$. For each such vertex, we replace the first symbol a_2 by one of the possible symbols a'_2 such that now $a'_2 \neq a_3, a_{\ell+1}$ (so $a'_2 \ldots a_{\ell+1}$ represents an allowed vertex). Then, we add arcs from vertex $a_1 \ldots a_\ell$ to vertex $a'_2 \ldots a_{\ell+1}$, with $a_1 \neq a_\ell$ and $a'_2 \neq a_3, a_{\ell+1}$. Note that $CK(d, \ell)$ and $MCK(d, \ell)$ have the same vertices, because we only add arcs to $CK(d, \ell)$ to obtain $MCK(d, \ell)$. See a

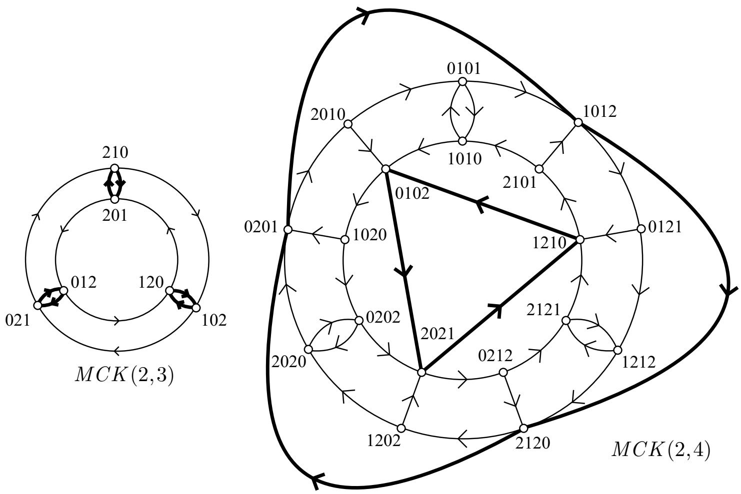


Fig. 3. Modified cyclic Kautz digraphs $MCK(2,3)$ and $MCK(2,4)$ (the thick lines are the arcs added with respect to the corresponding cyclic Kautz digraphs).

pair of examples of modified cyclic Kautz digraphs in Figure 3.

Theorem 3.1 *The modified cyclic Kautz digraph $MCK(d,\ell)$ has the following properties:*

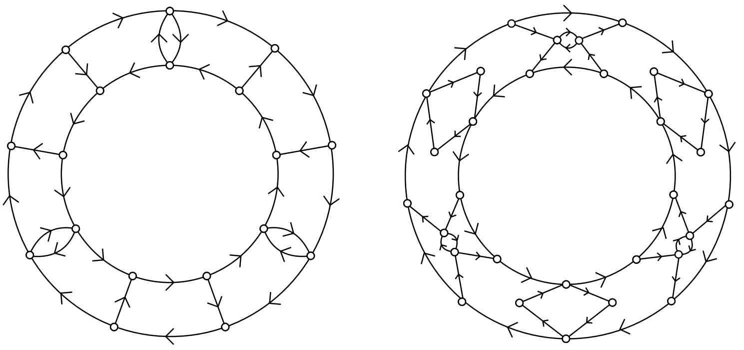
(a) *It is d-out-regular.*

(b) *Its diameter is $D = \ell$, which is the same as the diameter of the Kautz digraph $K(d,\ell)$.*

4 Line digraphs iterations of the cyclic Kautz digraphs

As it was done with the Kautz digraphs $K(d,\ell)$, which are regular as said before, here we compute the number of vertices of the t-iterated line digraphs $L^t(CK(d,\ell))$ of the cyclic Kautz digraphs $CK(d,\ell)$, which are non-regular digraphs. In contrast with the regular digraphs, the resolution of the non-regular case is not immediate. Indeed, this is an interesting combinatorial problem also for other non-regular digraphs. The diameter of a line digraph $L(G)$ of a digraph G is $D(L(G)) = D(G) + 1$, even if G is a non-regular digraph, with the exception of directed cycles (see Fiol, Yebra and Alegre [2]). Then, with the line digraph technique, we obtain digraphs with small diameter, small out-degrees and large number of vertices. Figure 4 shows an example of a $CK(d,\ell)$ and its line digraph. We present the following result.

Theorem 4.1 *Let $\ell \geq 3$ and $d \geq 1$ be integers. Then the number of vertices*



$CK(2,4)$ $\qquad$ $L(CK(2,4))$

Fig. 4. $CK(2,4)$ and its line digraph at iteration $t=1$.

of the iterated line digraph of $CK(d,\ell)$ *at iteration* t*, for* $1 \le t \le \ell-2$*, is*

$$(d^2-d+1)^t d^{\ell-t} + \frac{1}{2}(-1)^{\ell+1}(d-2)^t(d-1)d + \frac{1}{2}(-1)^{\ell} d^{t+1}(d+1).$$

Recall that a vertex of $CK(d,\ell)$ is a sequence of ℓ characters from an alphabet of $d+1$ symbols, such that consecutive symbols, and also the first and last symbols, are different. Two vertices of $CK(d,\ell)$ are adjacent when they have the form $a_1a_2\ldots a_\ell$ and $a_2\ldots a_\ell a_{\ell+1}$ (with $a_1 \neq a_\ell$ and $a_2 \neq a_{\ell+1}$). This suggests to represent an arc of $CK(d,\ell)$ as a sequence of $\ell+1$ characters $a_1a_2\ldots a_\ell a_{\ell+1}$ satisfying $a_i \neq a_{i+1}$ for $1 \le i \le \ell$, $a_1 \neq a_\ell$, and $a_2 \neq a_{\ell+1}$. Note that a_1 can be equal to $a_{\ell+1}$. The arcs of $CK(d,\ell)$ are the vertices of the iterated line digraph of $CK(d,\ell)$ at the first iteration $t=1$. Two such vertices are adjacent when they have the form $a_1a_2\ldots a_\ell a_{\ell+1}$ and $a_2a_3\ldots a_{\ell+1}a_{\ell+2}$, with $a_1 \neq a_\ell$, $a_2 \neq a_{\ell+1}$ and $a_3 \neq a_{\ell+2}$. Therefore, a vertex of the iterated line digraph of a cyclic Kautz digraph $CK(d,\ell)$ at iteration $t=2$ can be represented by a sequence of $\ell+2$ characters satisfying $a_i \neq a_{i+1}$, $a_1 \neq a_\ell$, $a_2 \neq a_{\ell+1}$ and $a_3 \neq a_{\ell+2}$. In general, for $0 \le t \le \ell-2$, the vertices of the iterated line digraph of a cyclic Kautz digraph $CK(d,\ell)$ at iteration t are represented by sequences $a_1a_2\ldots a_{\ell+t}$ satisfying $a_i \neq a_{i+1}$ for $1 \le i \le \ell+t-1$ and $a_i \neq a_{i+\ell-1}$ for $1 \le i \le t+1$. With these considerations, we obtain a system of recurrence equations for the sequences, which represent the numbers of vertices of $L^t(CK(d,\ell))$ in all the possible cases. The solution of this system of recurrences gives us the result of this last theorem.

Note that, in general, the t-iterated line digraph of a cyclic Kautz digraph is neither a Kautz digraph, nor a cyclic Kautz digraph. But if the length is $\ell=2$, then it is clear that $CK(d,2)$ (and all its iterated line digraphs) are Kautz digraphs.

Theorem 4.1 gives the number of vertices at the t-iteration of the line digraph of $CK(d,\ell)$, with $1 \le t \le \ell - 2$. Now we compute the number of vertices of $L^t(CK(d,\ell))$ without restriction on the value of t, for the particular case $\ell = 4$. Let $N_t = |L^t(CK(d,4))|$.

Proposition 4.2 *The number of vertices N_t of the iterated line digraph of $CK(d,4)$ at iteration $t \ge 0$ is*

$$N_t = \alpha\left(\frac{d-1+\sqrt{d^2-2d+5}}{2}\right)^t + \beta\left(\frac{d-1-\sqrt{d^2-2d+5}}{2}\right)^t,$$

where $\alpha = \frac{1}{2}d(d+1)\left(d^2-d+1+\frac{d^3-2d^2+4d-1}{\sqrt{d^2-2d+5}}\right)$ and $\beta = \frac{1}{2}d(d+1)\Big(d^2-d+1$ $-\frac{d^3-2d^2+4d-1}{\sqrt{d^2-2d+5}}\Big)$. Moreover, if $d=2$, N_t follows a Fibonacci sequence with initial values 18 *and* 30*.*

We leave as open problems the following two questions: Find the number of vertices of the t-iterated cyclic Kautz digraph $CK(d,\ell)$ for the remaining values of d, t and ℓ; and compute the number of vertices of the t-iterated modified cyclic Kautz digraph $MCK(d,\ell)$.

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