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Bipartite Kneser graphs are Hamiltonian

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Abstract

The Kneser graph $K(n,k)$ has as vertices all k-element subsets of $[n] := \{1,2,\ldots,n\}$ and an edge between any two vertices (=sets) that are disjoint. The bipartite Kneser graph $H(n,k)$ has as vertices all k-element and $(n-k)$-element subsets of $[n]$ and an edge between any two vertices where one is a subset of the other. It has long been conjectured that all connected Kneser graphs and bipartite Kneser graphs (apart from few trivial exceptions) have a Hamilton cycle. The main contribution of this work is proving this conjecture for bipartite Kneser graphs. We also establish the existence of long cycles in Kneser graphs (visiting almost all vertices), generalizing and improving upon previous results on this problem.

Keywords: Hamilton cycle, Kneser graph, hypercube, vertex-transitive graph

1 Introduction

The question whether a graph has a Hamilton cycle — a cycle that visits every vertex exactly once — is a fundamental graph theoretical problem with a wide range of practical applications, shown to be NP-complete already in

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Karp's landmark paper [19]. As a consequence, recent years have seen an increasing interest in Hamiltonicity problems in various different flavors and the solution of several long-standing open problems (the survey [20] gives an excellent overview over these developments).

1.1 Hamilton cycles in (bipartite) Kneser graphs

The question whether a graph has a Hamilton cycle turns out to be surprisingly difficult even for families of graphs defined by very simple algebraic constructions. Two prominent examples of this phenomenon are the Kneser graph and the bipartite Kneser graph (Kneser graphs were introduced by Lovász in his celebrated proof of Kneser's conjecture [23]). For integers n and k satisfying $k \geq 1$ and $n \geq 2k+1$, the *Kneser graph* $K(n,k)$ has as vertices all k-element subsets of $[n] := \{1,2,\ldots,n\}$, and an edge between any two vertices (=sets) that are disjoint. The *bipartite Kneser graph* $H(n,k)$ has as vertices all k-element and all $(n-k)$-element subsets of $[n]$, and an edge between any two vertices where one is a subset of the other. The Kneser graphs and bipartite Kneser graphs have long been conjectured to have a Hamilton cycle, apart from one notorious exception, namely the Petersen graph $K(5,2)$:

Conjecture 1.1 *For any $k \geq 1$ and $n \geq 2k+1$, except for $(n,k) = (5,2)$, the Kneser graph $K(n,k)$ has a Hamilton cycle.*

Conjecture 1.2 *For any $k \geq 1$ and $n \geq 2k+1$, the bipartite Kneser graph $H(n,k)$ has a Hamilton cycle.*

In the numerous papers on the subject (see below), the sparsest among these graphs, the so-called *odd graph* $K(2k+1,k)$ and the *middle layer graph* $H(2k+1,k)$ have received particular attention, as proving Hamiltonicity for the sparsest graphs is particularly intricate:

Conjecture 1.3 *For any $k \geq 1$, except for $k = 2$, the odd graph $K(2k+1,k)$ has a Hamilton cycle.*

Conjecture 1.4 *For any $k \geq 1$, the middle layer graph $H(2k+1,k)$ has a Hamilton cycle.*

One of the main motivations for these conjectures is a classical and vastly more general conjecture due to Lovász [22], which asserts that, apart from five exceptional graphs (one of the exceptions $K(5,2)$ we already mentioned), every connected vertex-transitive graph has a Hamilton cycle. A vertex-transitive graph is a graph that 'looks the same' from the point of view of any vertex, and Kneser graphs and bipartite Kneser graphs have this strong symmetry property (and they are connected for the given range of parameters), so these conjectures represent a highly nontrivial special case of Lovász' conjecture.

1.2 Known results

Conjecture 1.3 was raised by Meredith and Lloyd [25] (see also [1]). In a sequence of papers [3,4,6,18,14,24,25], the conjecture and its generalization, Conjecture 1.1, were verified for ever increasing ranges of parameters. To date, Conjecture 1.1 has been confirmed with the help of computers for all $n \leq 27$ and all relevant values of k [30], and the best known general result is due to Chen:

Theorem 1.5 ([5]) *For any $k \geq 1$ and $n \geq 2.62k + 1$, the Kneser graph $K(n, k)$ has a Hamilton cycle.*

As an important step towards settling Conjecture 1.3, Johnson showed that the odd graph contains a cycle that visits almost all vertices:

Theorem 1.6 ([17]) *There exists a constant c, such that for any $k \geq 1$, the odd graph $K(2k + 1, k)$ has a cycle that visits at least a $(1 - \frac{c}{\sqrt{k}})$-fraction of all vertices.*

Conjecture 1.2 was raised independently by Simpson [33] and Roth (see [10] and [16]). Since then, there has been steady progress on the problem [4,16,34], and similarly to before, the conjecture has been confirmed for all $n \leq 27$ and all relevant values of k [30], and the best known general result is due to Chen:

Theorem 1.7 ([5]) *For any $k \geq 1$ and $n \geq 2.62k + 1$, the bipartite Kneser graph $H(n, k)$ has a Hamilton cycle.*

Conjecture 1.4, also known as *middle levels conjecture* or *revolving door conjecture*, originated probably with Havel [13] and Buck and Wiedemann [2], but has also been attributed to Dejter, Erdős, Trotter [21] and various others. This conjecture has attracted considerable attention over the years (see e.g. [7,8,9,11,15,17,21,27,29,31,32]), and a proof of it has only been announced very recently.

Theorem 1.8 ([26]) *For any $k \geq 1$, the middle layer graph $H(2k+1, k)$ has a Hamilton cycle.*

1.3 Hamilton cycles in the hypercube

The main reason for the interest in the middle levels conjecture is its relation to the hypercube graph and to Gray codes, two themes of fundamental interest for combinatorialists (see the surveys [12] and [28], respectively). The *hypercube* $Q(n)$ is the graph which has as vertices all bitstrings of length n, and an edge between any two bitstrings that differ in exactly one bit. Partitioning the vertices of $Q(n)$ into *levels* $0,\ldots,n$ according to the number of 1-entries in the bitstrings, and denoting by $Q(n,k)$ the subgraph of $Q(n)$ induced by all vertices in level k and $k+1$, it is easy to see that $H(2k+1,k)$ and $Q(2k+1,k)$ are isomorphic. So the middle levels conjecture asserts that the subgraph $Q(2k+1,k)$ of the cube has a Hamilton cycle. Observe that Hamilton cycles in the cube or subgraphs of it correspond to certain *Gray codes*, i.e., cyclic sequences of binary code words with the property that any two consecutive code words differ in exactly one bit. Clearly, $Q(2k+1,k)$ is the only subgraph of the cube induced by two consecutive levels that have the same size, and where we can hope to find a Hamilton cycle. Nevertheless, the following is a natural generalization of the middle levels conjecture (in a different direction than Conjecture 1.2, cf. also [11]), which provides a nice structural insight about the cube and establishes the existence of various additional families of restricted Gray codes:

Theorem 1.9 *For any $n\geq 3$ and $k\in\{1,2,\ldots,n-2\}$, the graph $Q(n,k)$ has a cycle that visits all vertices in the smaller of the levels k and $k+1$.*

It was already noted in [13] that with a simple inductive construction, Theorem 1.9 can be derived easily from Theorem 1.8. In fact, the results presented in this extended abstract are proved using a further refinement of this proof technique.

2 Our results

The main contribution of this work is a proof of Conjecture 1.2.

Theorem 2.1 *For any $k\geq 1$ and $n\geq 2k+1$, the bipartite Kneser graph $H(n,k)$ has a Hamilton cycle.*

We also make some progress towards Conjecture 1.1 (and the special case Conjecture 1.3), by generalizing and improving Theorem 1.6 as follows:

Theorem 2.2 *For any $k\geq 1$ and $n\geq 2k+1$, the Kneser graph $K(n,k)$ has a cycle that visits at least a $\frac{2k}{n}$-fraction of all vertices. In particular, for any*

$k \geq 1$, *the odd graph* $K(2k+1,k)$ *has a cycle that visits at least a* $(1-\frac{1}{2k+1})$-*fraction of all vertices.*

Note that the cycle guaranteed by Theorem 2.2 visits almost all vertices of $K(n,k)$, i.e., a $(1-o(1))$-fraction, whenever $n = 2k + o(k)$.

3 Proof ideas

Our results are immediate consequences of the following lemma, illustrated in Figure 1 below. This lemma therefore represents a powerful 'bootstrapping' method that extends Theorem 1.8 to a large range of other interesting graphs. To state the lemma, we say that a path in the hypercube $Q(n)$ is *monotone*, if it visits at most one vertex in every level.

Lemma 3.1 *For any* $k \geq 1$ *and* $n \geq 2k+1$, *there is a cycle* $C(n,k)$ *of length* $2\binom{n}{k}$ *in the graph* $Q(n,k) \subseteq Q(n)$ *that visits all vertices in level* k, *and a set of* $\binom{n}{k}$ *many vertex-disjoint monotone paths* $\mathcal{P}(n,k)$ *in* $Q(n)$, *each of which starts at a vertex of the cycle* $C(n,k)$ *in level* $k+1$ *and ends at a vertex in level* $n-k$.

Note that Lemma 3.1 is a strengthening of Theorem 1.9 (for the theorem, the paths $\mathcal{P}(n,k)$ are ignored, and the cycle $C(n,k)$ alone has the desired properties).

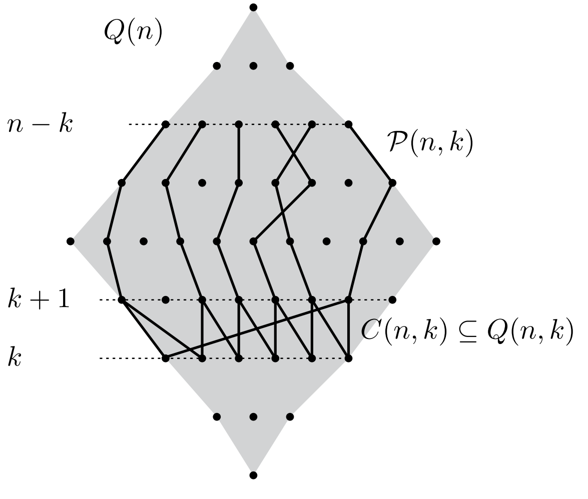

Fig. 1. Illustration of Lemma 3.1.

In this extended abstract we only sketch the proof of Lemma 3.1: The proof proceeds by a straightforward induction: The cycle $C(n,k)$ and the corresponding paths $\mathcal{P}(n,k)$ are constructed from $C(n-1,k)$, $\mathcal{P}(n-1,k)$ and from $C(n-1,k-1)$ and $\mathcal{P}(n-1,k-1)$ (for this the induction hypothesis needs to be strengthened slightly, by enforcing and forbidding certain vertices to be visited by the subgraphs $C(n,k)$ and $\mathcal{P}(n,k)$). One base case of the induction $(k,n)=(1,n)$ is easily verified 'manually', and the other base case $(k,n)=(k,2k+1)$ is exactly the middle levels conjecture, which we know to be true by Theorem 1.8.

With Lemma 3.1 in hand, proving Theorems 2.1 and 2.2 is easy.

Proof of Theorem 2.1. Let n and k be as in the theorem, and let $C(n,k)$ and $\mathcal{P}(n,k)$ be the cycle and the set of paths given by Lemma 3.1. The cycle $C(n,k)$ visits all $N := \binom{n}{k}$ vertices in level k, and it has the form $(x_1, x_2, \ldots, x_{2N})$, where the x_{2i-1} and the x_{2i}, $i = 1, \ldots, N$, are vertices in level k and level $k+1$, respectively. Moreover, every x_{2i} is obtained from x_{2i-1} or from x_{2i+1} (indices are considered modulo $2N$) by flipping a single 0-bit to a 1-bit. For $i = 1, \ldots, N$ consider the path from $\mathcal{P}(n,k)$ whose first vertex is x_{2i}, and let y_{2i} be its end vertex in level $n-k$. As the path is monotone, y_{2i} is obtained from x_{2i} by flipping $(n-k)-(k+1) = n-2k-1$ many 0-bits to 1-bits. Now consider the cyclic sequence $(x_1, y_2, x_3, y_4, x_5, y_6, \ldots, x_{2N-1}, y_{2N})$ of vertices. Note that the vertices $\{x_{2i-1} \mid i = 1, \ldots, N\}$, are all vertices in level k, the vertices $\{y_{2i} \mid i = 1, \ldots, N\}$ are all vertices in level $n-k$ (the paths from $\mathcal{P}(n,k)$ are vertex-disjoint). Moreover, every y_{2i} is obtained from x_{2i-1} or from x_{2i+1} by flipping $n-2k$ many 0-bits to 1-bits. Interpreting the bitstrings in this sequence as characteristic vectors of subsets of $[n]$, we thus obtain the desired Hamilton cycle in $H(n,k)$. □

Proof of Theorem 2.2. For $k = 1$ and $n \geq 3$ the graph $K(n,1)$ is the complete graph on n vertices and trivially has a Hamilton cycle. So let $k \geq 2$ and $n \geq 2k+1$, and let $C(n-1,k-1)$ and $\mathcal{P}(n-1,k-1)$ be the cycle and the set of paths given by Lemma 3.1. The paths in $\mathcal{P}(n-1,k-1)$ start in level k and end in level $(n-1)-(k-1) = n-k$, and therefore have length $n-2k \geq 1$. The cycle $C(n-1,k-1)$ visits all $N := \binom{n-1}{k-1}$ vertices in level $k-1$, and it has the form $(x_1, x_2, \ldots, x_{2N})$, where the x_{2i-1} and the x_{2i}, $i = 1, \ldots, N$, are vertices in level $k-1$ and level k, respectively. Moreover, every x_{2i} is obtained from x_{2i-1} or from x_{2i+1} (indices are considered modulo $2N$) by flipping a single 0-bit to a 1-bit. For $i = 1, \ldots, N$ consider the path from $\mathcal{P}(n-1,k-1)$ whose first vertex is x_{2i}, and let y_{2i} be the vertex of this path in level $n-k-1$ (the end vertex of this path is on the next higher level $n-k$). As the path is monotone, y_{2i} is obtained from x_{2i} by flipping $(n-k-1)-k = n-2k-1$ many 0-bits to 1-bits. For $i = 1, \ldots, N$, let x_{2i-1}^+ be the bitstring obtained from x_{2i-1}

by adding an additional 1-bit, and let $\overline{y_{2i}}^+$ be the bitstring obtained from y_{2i} by inverting all bits and adding an additional 0-bit. Note that x_{2i-1}^+ and $\overline{y_{2i}}^+$ both have length n and contain exactly k entries equal to 1. Now consider the cyclic sequence of vertices $(x_1^+, \overline{y_2}^+, x_3^+, \overline{y_4}^+, x_5^+, \overline{y_6}^+, \ldots, x_{2N-1}^+, \overline{y_{2N}}^+)$. Note that all vertices in this sequence are different (here we use that the y_{2i} are all different, as the paths from $\mathcal{P}(n-1, k-1)$ are vertex-disjoint). Moreover, for every $\overline{y_{2i}}^+$ we have that at each position with a 1-bit, both x_{2i-1}^+ and x_{2i+1}^+ have a 0-bit. Interpreting the bitstrings in this sequence as characteristic vectors of subsets of $[n]$, we thus obtain a cycle of length $2N = 2\binom{n-1}{k-1}$ in $K(n,k)$. The total number of vertices of $K(n,k)$ is $\binom{n}{k}$, so the fraction of vertices visited by the cycle is $2\binom{n-1}{k-1}/\binom{n}{k} = \frac{2k}{n}$. □

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