Kempe Equivalence of Colourings of Cubic Graphs

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Abstract

Given a graph $G = (V, E)$ and a proper vertex colouring of G, a Kempe chain is a subset of V that induces a maximal connected subgraph of G in which every vertex has one of two colours. To make a Kempe change is to obtain one colouring from another by exchanging the colours of vertices in a Kempe chain. Two colourings are Kempe equivalent if each can be obtained from the other by a series of Kempe changes. A conjecture of Mohar asserts that, for $k \geq 3$, all k-colourings of connected k-regular graphs that are not complete are Kempe equivalent. We address the case $k = 3$ by showing that all 3-colourings of a connected cubic graph G are Kempe equivalent unless G is the complete graph K_4 or the triangular prism.

Keywords: Kempe equivalence, cubic graph, graph colouring.

1 Introduction

Let $G = (V, E)$ denote a simple undirected graph and let k be a positive integer. A *k-colouring* of G is a mapping $\phi : V \to \{1, \dots, k\}$ such that $\phi(u) = \phi(v)$ if $uv \in E$. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest k such that G has a k-colouring.

If a and b are distinct colours, then $G(a, b)$ denotes the subgraph of G induced by vertices with colour a or b. An (a, b)-*component* of G is a connected

component of $G(a, b)$ and is known as a *Kempe chain*. A *Kempe change* is the operation of interchanging the colours of some (a, b)-component of G. Let $C_k(G)$ be the set of all k-colourings of G. Two colourings $\alpha, \beta \in C_k(G)$ are *Kempe equivalent*, denoted by $\alpha \sim_k \beta$, if each can be obtained from the other by a series of Kempe changes. The equivalence classes $C_k(G)/ \sim_k$ are called *Kempe classes*.

Kempe changes were first introduced by Kempe in his well-known failed attempt at proving the Four-Colour Theorem. The Kempe change method has proved to be a powerful tool with applications to several areas such as timetables [16], theoretical physics [20,21], and Markov chains [19]. The reader is referred to [15,17] for further details. From a theoretical viewpoint, Kempe equivalence was first addressed by Fisk [10] who proved that all 4-colourings of an Eulerian triangulation of the plane are Kempe equivalent. This result was later extended by Meyniel [13] who showed that all 5-colourings of a planar graph are Kempe equivalent, and by Mohar [15] who proved that all k-colourings, $k > \chi(G)$, of a planar graph G are Kempe equivalent. Las Vergnas and Meyniel [18] extended Meyniel's result by proving that all 5-colourings of a K_5-minor free graph are Kempe equivalent. Bertschi [2] also showed that all k-colourings of a perfectly contractile graph are Kempe equivalent, thus answering a conjecture of Meyniel [14]. We note that Kempe equivalence with respect to edge-colourings has also been investigated [1,12,15].

Here we are concerned with a conjecture of Mohar [15] on connected k-*regular* graphs, that is, graphs in which every vertex has degree k for some $k \geq 0$. Note that, for every connected 2-regular graph G that is not an odd cycle, it holds that $C_2(G)$ is a Kempe class. Mohar conjectured the following (where K_{k+1} is the complete graph on $k + 1$ vertices).

Conjecture 1.1 ([15]) *Let $k \geq 3$. If G is a connected k-regular graph that is not K_{k+1} then $C_k(G)$ is a Kempe class.*

Excluding the complete graph K_{k+1} in Conjecture 1.1 is a necessary condition as $\chi(K_{k+1}) = k + 1$. In all other cases, it follows by Brooks' Theorem [6] that every connected k-regular graph that is not complete nor an odd cycle has a k-colouring. We address Conjecture 1.1 for the case $k = 3$. For this case the conjecture is known to be false. A counterexample is the 3-prism displayed in Figure 1 (along with a claw that will appear later). The fact that some 3-colourings of the 3-prism are not Kempe equivalent was already observed by van den Heuvel [11]. Our contribution is that the 3-prism is the *only* counterexample for the case $k = 3$; that is, we completely settle the case $k = 3$ by proving the following result for 3-regular graphs also known as *cubic* graphs.

Fig. 1. A claw and a 3-prism.

Theorem 1.2 *If G is a connected cubic graph that is neither K_4 nor the 3-prism then $C_3(G)$ is a Kempe class.*

We sketch the proof of our result in the next section. Besides exploiting the 3-regularity, our proof also takes into account the fact that one additional forbidden graph, namely the 3-prism, is forbidden. We did not find any counterexamples for $k \geq 4$ and believe Conjecture 1.1 may well hold for $k \geq 4$. As such, new techniques are necessary to tackle the remaining cases.

Our result is an example of a type of result that has received much recent attention: that of determining the structure of a *reconfiguration graph*. A reconfiguration graph has as vertex set all solutions to a search problem and an edge relation that describes a transformation of one solution into another. Thus Theorem 1.2 is concerned with the reconfiguration graph of 3-colourings of cubic graph with edge relation $\sim_k$ and shows that it is connected except in two cases. To date the stucture of reconfiguration graphs of colourings has focused [3–5, 7–9] on the case where vertices are joined by an edge only when they differ on just one colour (that is, when one colouring can be transformed into another by a Kempe change of a Kempe chain that contains only one vertex). For a survey of recent results on reconfiguration graphs, see [11].

2   The Proof of Theorem 1.2

We first give some further definitions and terminology. Let $G = (V, E)$ be a graph. Then G is *H-free* for some graph H if G does not contain an induced subgraph isomorphic to H. A *separator* of G is a set $S \subset V$ such that $G - S$ has more components than G. We say that G is *p-connected* for some integer p if $|V| \geq p + 1$ and every separator of G has size at least p. Some small graphs that we will refer to are defined by their illustrations in Figure 1.

Besides three new lemmas, we will need the aforementioned result of van den Heuvel, which follows from the fact that for the 3-prism T, the subgraphs $T(1, 2)$, $T(2, 3)$ and $T(1, 3)$ are connected so that the number of Kempe classes is equal to the number of different 3-colourings of T up to colour permutation, which is two.

Lemma 2.1 ([11]) *If G is the 3-prism then $C_3(G)$ consists of two Kempe classes.*

Lemma 2.2 *If G is a connected cubic graph that is not 3-connected then $C_3(G)$ is a Kempe class.*

Lemma 2.3 *If G is a 3-connected cubic graph that is claw-free but that is neither K_4 nor the 3-prism then $C_3(G)$ is a Kempe class.*

Lemma 2.4 *If G is a 3-connected cubic graph that is not claw-free then $C_3(G)$ is a Kempe class.*

Observe that Theorem 1.2 follows from the above lemmas, which form a case distinction. Hence it suffices to prove Lemmas 2.2–2.4. We prove Lemma 2.2 in the remainder [5].

We need three auxiliary results and one more definition: a graph G is *d-degenerate* if every induced subgraph of G has a vertex with degree at most d.

Lemma 2.5 ([15, 18]) *Let d and k be any two integers with $d \geq 0$ and $k \geq d+1$. If G is a d-degenerate graph then $C_k(G)$ is a Kempe class.*

Lemma 2.6 ([18]) *Let $k \geq 1$ be an integer. Let G_1, G_2 be two graphs such that $G_1 \cap G_2$ is complete. If both $C_k(G_1)$ and $C_k(G_2)$ are Kempe classes then $C_k(G_1 \cup G_2)$ is a Kempe class.*

Lemma 2.7 ([15]) *Let $k \geq 1$ be an integer and let G be a subgraph of a graph G'. Let c_1 and c_2 be the restrictions, to G, of two k-colourings c'_1 and c'_2 of G'. If c'_1 and c'_2 are Kempe equivalent then c_1 and c_2 are Kempe equivalent.*

Proof of Lemma 2.2. Let G be a connected cubic graph that is not 3-connected. As G is cubic, G has at least four vertices. Because G is not 3-connected, G has a separator S of size at most 2. Let S be a minimum separator of G such that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = S$. As every vertex in S has degree at most 2 in each G_i and G is cubic, G_1 and G_2 are 2-degenerate. Hence, by Lemma 2.5, $C_3(G_1)$ and $C_3(G_2)$ are Kempe classes. If S is a clique, we apply Lemma 2.6. Thus we assume that S, and any other minimum separator of G, is not a clique. Then $S = \{x, y\}$ for two distinct vertices x and y with $xy \notin E(G)$.

Because S is a minimum separator, x and y are non-adjacent and G is cubic, x has either one neighbour in G_1 and two in G_2, or the other way around; the same holds for vertex y. For $i = 1, 2$, let $N_i(x)$ and $N_i(y)$ be the

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[5] A manuscript containing full proofs is available from arxiv.org/abs/1503.03430

set of neighbours of x and y, respectively, in G_i. Then we have that either $|N_1(x)| = 1$ and $|N_2(x)| = 2$, or $|N_1(x)| = 2$ and $|N_2(x)| = 1$, and similarly, that either $|N_1(y)| = 1$ and $|N_2(y)| = 2$, or $|N_1(y)| = 2$ and $|N_2(y)| = 1$. Let $x_1 \in N_1(x)$ for some $x_1 \in V(G_1)$.

We may assume that $|N_1(x)| \neq |N_1(y)|$; if not we can do as follows. Assume without loss of generality that $N_1(x) = \{x_1\}$ and that $|N_1(y)| = 1$. Then $\{x_1, y\}$ is a separator. By our assumption that G has no minimum separator that is a clique, we find that $\{x_1, y\}$ is a minimum separator with $x_1y \notin E(G)$. As G is cubic, x_1 has two neighbours in $V(G_1) \setminus \{x, x_1\}$. As $|N_1(y)| = 1$ and x_1 and y are not adjacent, y has exactly one neighbour in $V(G_1) \setminus \{x, x_1\}$. Hence we could take $\{x_1, y\}$ as our minimum separator instead of S in order to get the desired property. We may thus assume that $|N_1(x)| \neq |N_1(y)|$. As this means that $|N_2(x)| \neq |N_2(y)|$, we can let $N_1(x) = \{x_1\}$ and $N_2(y) = \{y_1\}$ for some $y_1 \in V(G_2)$.

It now suffices to prove the following two claims.

Claim 1 *All colourings α such that $\alpha(x) \neq \alpha(y)$ are Kempe equivalent in $C_3(G)$.*

We prove Claim 1 as follows. We add an edge e between x and y. This results in graphs $G_1 + e$, $G_2 + e$ and $G + e$. We first prove that $C_3(G + e)$ is a Kempe class. Because x and y have degree 1 in G_1 and G_2, respectively, and G is cubic, we find that the graphs $G_1 + e$ and $G_2 + e$ are 2-degenerate. Hence, by Lemma 2.5, $C_3(G_1 + e)$ and $C_3(G_2 + e)$ are Kempe classes. By Lemma 2.6, it holds that $C_3(G + e)$ is a Kempe class. Applying Lemma 2.7 completes the proof of Claim 1.

Claim 2 *For every colouring α such that $\alpha(x) = \alpha(y)$, there exists a colouring β with $\beta(x) \neq \beta(y)$ such that α and β are Kempe equivalent in $C_3(G)$.*

We assume without loss of generality that $\alpha(x) = \alpha(y) = 1$ and $\alpha(y_1) = 2$. If $\alpha(x_1) = 2$ then we apply a Kempe change on the $(1, 3)$-component of G that contains x. Note that y does not belong to this component. Hence afterwards we obtain the desired colouring γ. If $\alpha(x_1) = 3$ then we first apply a Kempe change on the $(2, 3)$-component of G that contains x_1. Note that this does not affect the colours of x, y and y_1 as they do not belong to this component. Afterwards we proceed as before. This completes the proof of Claim 2 (and the lemma). $\qquad\square$

References

[1] S.-M. Belcastro and R. Haas. Counting edge-kempe-equivalence classes for 3-edge-colored cubic graphs. *Discrete Mathematics*, 325(0):77 – 84, 2014.

[2] M. E. Bertschi. Perfectly contractile graphs. *Journal of Combinatorial Theory, Series B*, 50(2):222 – 230, 1990.

[3] M. Bonamy and N. Bousquet. Recoloring bounded treewidth graphs. In *Proc. LAGOS 2013, Electronic Notes in Discrete Mathematics*, volume 44, pages 257–262, 2013.

[4] M. Bonamy, M. Johnson, I. M. Lignos, V. Patel, and D. Paulusma. Reconfiguration graphs for vertex colourings of chordal and chordal bipartite graphs. *Journal of Combinatorial Optimization*, 27:132–143, 2014.

[5] N. Bousquet and G. Perarnau. Fast recoloring of sparse graphs. *arXiv*, 1411.6997, 2014.

[6] R. L. Brooks. On colouring the nodes of a network. *Mathematical Proceedings of the Cambridge Philosophical Society*, 37:194–197, 1941.

[7] L. Cereceda, J. van den Heuvel, and M. Johnson. Connectedness of the graph of vertex-colourings. *Discrete Mathematics*, 308:913–919, 2008.

[8] L. Cereceda, J. van den Heuvel, and M. Johnson. Finding paths between 3-colorings. *Journal of Graph Theory*, 67(1):69–82, 2011.

[9] C. Feghali, M. Johnson, and D. Paulusma. A reconfigurations analogue of Brooks' theorem. In *Proceedings MFCS 2014*, volume 8635 of *LNCS*, pages 287–298. Springer, 2014.

[10] S. Fisk. Geometric coloring theory. *Advances in Mathematics*, 24(3):298 – 340, 1977.

[11] J. van den Heuvel. The complexity of change. *Surveys in Combinatorics, London Mathematical Society Lecture Notes Series*, 409, 2013.

[12] J. McDonald, B. Mohar, and D. Scheide. Kempe equivalence of edge-colorings in subcubic and subquartic graphs. *Journal of Graph Theory*, 70(2):226–239, 2012.

[13] H. Meyniel. Les 5-colorations d'un graphe planaire forment une classe de commutation unique. *Journal of Combinatorial Theory, Series B*, 24(3):251 – 257, 1978.

[14] H. Meyniel. The graphs whose odd cycles have at least two chords. *Topics on Perfect Graphs*, 21:115–120, 1984.

[15] B. Mohar. Kempe equivalence of colorings. In *Graph Theory in Paris*, Trends in Mathematics, pages 287–297. Birkhuser Basel, 2007.

[16] M. Mühlenthaler and R. Wanka. The connectedness of clash-free timetables. In *Proceedings PATAT 2014*, pages 330–346, 2014.

[17] A. Sokal. A personal list of unsolved problems concerning lattice gasses and antiferromagnetic potts models. *Markov Process. Related Fields*, 7:21–38, 2000.

[18] M. L. Vergnas and H. Meyniel. Kempe classes and the hadwiger conjecture. *Journal of Combinatorial Theory, Series B*, 31(1):95 – 104, 1981.

[19] E. Vigoda. Improved bounds for sampling colorings. *Journal of Mathematical Physics*, 41(3), 2000.

[20] J.-S. Wang, R. H. Swendsen, and R. Kotecký. Antiferromagnetic potts models. *Phys. Rev. Lett.*, 63:109–112, Jul 1989.

[21] J.-S. Wang, R. H. Swendsen, and R. Kotecký. Three-state antiferromagnetic potts models: A monte carlo study. *Phys. Rev. B*, 42:2465–2474, Aug 1990.