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An Improved Upper Bound on the Growth Constant of Polyominoes

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Abstract

Polyominoes are edge-connected sets of squares on the square lattice. The symbol λ usually denotes the growth constant of $A(n)$, the sequence that enumerates polyominoes. In this paper we prove that $\lambda \leq 4.5685$ by analyzing the growth constant of a sequence $B(n)$, for which $B(n) \geq A(n)$ for any value of $n \in \mathbb{N}$. The recursive formula for $B(n)$ is based on the representation of a polyomino as the assembly of a pair of smaller polyominoes and a code that describes the assembly. Then, an upper bound on the growth constant of $B(n)$ is derived by a careful analysis of this assembly.

Keywords: Klarner's constant, asymptotic analysis.

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1 Introduction

A *polyomino* of size n is an edge-connected set of n cells on the square lattice $\mathbb{Z}^2$. Two *fixed* polyominoes are equivalent if one can be transformed into the other by a translation. In this paper we consider only fixed polyominoes, and so we simply refer to them in the sequel as "polyominoes." The study of polyominoes began in the 1950s in statistical physics [3,12], where they are usually called *lattice animals*. Counting polyominoes is a long-standing problem in enumerative combinatorics.

Let $A(n)$ denote the number of polyominoes of size n (sequence A001168 in the on-line encyclopedia of integer sequences [11]). Elements of the sequence $A(n)$ are currently known up to $n = 56$ [7]. The growth constant (asymptotic growth rate) of polyominoes has also attracted much attention in the literature. Klarner [8] showed that the limit $\lambda := \lim_{n\to\infty} \sqrt[n]{A(n)}$ exists. The convergence of $A(n+1)/A(n)$ to λ (as $n\to\infty$) was proven only three decades later by Madras [10] by using a novel pattern-frequency argument. The best-known lower bound on λ is 3.9801 [1].[3] It is widely *believed* (see, e.g., [5,6]) that $\lambda \approx 4.06$, and the currently best estimate, $\lambda = 4.0625696 \pm 0.0000005$, is due to Jensen [7].

In 1961, Eden [4] showed an upper bound on λ by using a simple argument: Any polyomino can be built by a set of $n - 1$ "instructions" taken from a superset of size $3(n - 1)$. (Some of these instruction sets are illegal, and some other sets produce the same polyomino, but this only helps.) Hence, $\lambda \le \lim_{n\to\infty} \binom{3(n-1)}{n-1}^{1/n} = 6.75$.

The currently best upper bound on λ was proven by Klarner and Rivest [9] in 1973 by generalizing Eden's method. Instead of simple "instructions" (telling how to add a single square to the polyomino) they used so-called "twigs" (composite instructions for adding several squares at once). By using larger and larger twigs (up to size 10), Klarner and Rivest were able to show that $\lambda \le 4.6496$. Since then, there were no developments with respect to the upper bound on λ.

In this paper we use a simple idea for improving the upper bound. We build a sequence $B(n)$ which is larger than $A(n)$ for almost every value of $n \in \mathbb{N}$. Thus, the growth constant of $B(n)$, if it exists, is in particular an upper bound on λ. We use this method for showing that $\lambda \le 4.5685$.

[3] In the informal note [2] it was shown that $\lambda > 4.0025$.

2 The Bound

Our first building block is the ability to partition every polyomino in a balanced manner.

Definition 2.1 A polyomino P can be *partitioned* into two polyominoes P_1, P_2 if the cell set of P can be split into two complementing non-empty subsets, such that each subset is a valid (connected) polyomino. We also say that the polyominoes P_1, P_2 can be *attached* to each other so as to yield the polyomino P.

"Attaching" polyominoes is a natural generalization of the widely-used notion *concatenating* polyominoes, that is, the operation of attaching them in a lexicographic order of their cells.

Definition 2.2 A partition of a polyomino of size n into two polyominoes P_1, P_2 is *k-balanced* if $k \le |P_i| \le n - k$ (for $i = 1, 2$).

Theorem 2.3 *Every polyomino of size n has at least one $\lceil (n-1)/4 \rceil$-balanced partition.*

Proof. (Sketch) Let us rephrase the claim in graph terminology. In fact, we prove a stronger claim which states that every connected graph G, for which $\Delta(G) \le 4$, can be partitioned into two vertex-disjoint subgraphs G_1, G_2, such that (1) G_1, G_2 are connected; and (2) $\lceil (n-1)/4 \rceil \le |G_i| \le \lfloor (3n+1)/4 \rfloor$ (for $i = 1, 2$). This can be done constructively by considering a spanning tree of G, marking an arbitrary vertex as its root, and traversing the tree downwards from the root while keeping the invariant that either the already-traversed subgraph meets the size requirement or the untraversed part contains a subgraph with this property. When the process stops, which must be the case, the desired partition is found. □

However, we do better than the above. In order to show that every polyomino has (at least) *two* balanced partitions, we very slightly reduce the balance parameter. Let

$$\pi(n) = \begin{cases} \lfloor (n-1)/4 \rfloor & \text{if } n \equiv 2 \pmod 4; \\ \lceil (n-1)/4 \rceil & \text{otherwise.} \end{cases}$$

Theorem 2.4 *Every polyomino of size n has at least two $\pi(n)$-balanced partitions.*

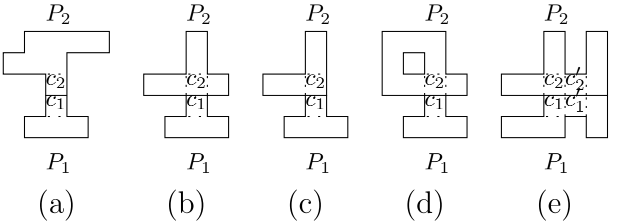

Fig. 1. Possible partitioning configurations (Theorem 2.4).

Proof. Let P be a polyomino of size n. By Theorem 2.3, we know that P has a $\lceil (n-1)/4 \rceil$-balanced partition into two polyominoes P_1, P_2 of total size n. Let P_1 be the smaller of the two polyominoes (that is, $|P_1| \le n/2$ and $|P_2| \ge n/2$), and let c_1, c_2 be cells of P_1, P_2, respectively, which touch each other when P_1, P_2 are attached and form P (see Fig. 1).

Let us show that P can be partitioned in a balanced manner in at least one more way. Consider the removal of c_2 from P_2, leaving between one to three connected components of P_2. (See a few options in Figs. 1(a-d).) Assume first that the size of at least one of these components, denoted as P_{2_a}, is at least $\lceil (n-1)/4 \rceil$. Let P_{2_b} be the remainder of P_2 (note that $c_2 \in P_{2_b}$). Then, the polyominoes $P_1 \cup P_{2_b}$ and P_{2_a} are also a $\lceil (n-1)/4 \rceil$-balanced partition of P.

Otherwise, the size of all components of P_2 is strictly less than $\lceil (n-1)/4 \rceil$. In this case there must be three components since the size of P_2 is at least $\lceil n/2 \rceil$. Consider now the number of components into which P_1 is split by the removal of c_1. If there is a single component (Fig. 1(b)), then the polyominoes $Q_1 = P_1 \setminus \{c_1\}$ and $P_2 \cup \{c_1\}$ are also a partition of P. However, in this particular case, the size of Q_1 may fall to $\lfloor (n-1)/4 \rfloor$. This happens when $3(\lceil (n-1)/4 \rceil - 1) + 1 = \lfloor (3n+1)/4 \rfloor$, which is the case iff $n \equiv 2 \pmod 4$.

Otherwise, we are in the case in which P_2 is split into three components (all of size less than $\lceil (n-1)/4 \rceil$) and P_1 is split into two or three components (Fig. 1(e)). This is the only case in which graph arguments do not suffice and a property of the lattice is needed. The crucial observation is that one of the neighbors of c_2 within P_2 (say, c_2') must touch one of the neighbors of c_1 (say, c_1')! We immediately obtain the alternative partition of P: Denote by P_{2_a} the component of P_2 that contains c_2'. The polyominoes $P_1 \cup P_{2_a}$ and $P_2 \setminus P_{2_a}$ are also a $\lceil (n-1)/4 \rceil$-balanced partition of P. This completes the proof. □

Unlike the concatenation operation, two polyominoes can be attached in many ways. We bound from above the number of attachments.

Theorem 2.5 *Let P_1, P_2 be two polyominoes of sizes n_1, n_2, respectively, s.t. $n_1 + n_2 = n$. Then, P_1, P_2 can be attached to each other in at most $2n$ ways.*

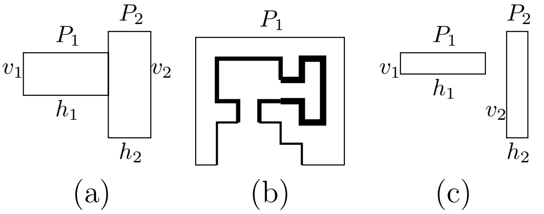

Fig. 2. Attachments of polyominoes.

Proof. (Sketch) A pair of rectangular polyominoes of dimensions $v_1 \times h_1$ and $v_2 \times h_2$ (Fig. 2(a)) can be attached in $2(v_1 + v_2 + h_1 + h_2 - 2)$ ways. A *cavity* in the boundary of a polyomino is a concave portion of it, and cavities can be nested (Fig. 2(b)). For a fixed perimeter, an induction on the level of cavity nesting shows that the maximum number of ways to attach the polyominoes is obtained when they have no cavities. (Hence, the maximum is neither obtained with polyominoes with holes.) In addition, for a specified size, the perimeter is maximized by tree polyominoes. The only tree polyominoes without cavities are "sticks" (Fig. 2(c)), for which $v_i + h_i = n_i + 1$, implying the claim. □

We can now prove our main result.

Theorem 2.6 $\lambda \le 4.5685$.

Proof. First, the combination of Theorems 2.4 and 2.5 implies that

$$A(n) \le n \sum_{k=\pi(n)}^{\lfloor n/2 \rfloor} A(k)A(n-k).$$

Indeed, every polyomino of size n can be partitioned in at least two $\pi(n)$-balanced ways into a pair of polyominoes P_1, P_2 of sizes n_1, n_2, respectively (where $n_1 + n_2 = n$), and a code with up to $2n$ options will tell us uniquely how to attach P_1, P_2 in order to reconstruct P. Naturally, P can be partitioned in more than two ways, and the number of options to attach P_1 and P_2 can be smaller than $2n$, but this only helps.

Second, define the sequence $B(n)$ as follows.

$$B(n) = \begin{cases} A(n) & 1 \le n \le 56; \\ n \sum_{k=\pi(n)}^{\lfloor n/2 \rfloor} B(k)B(n-k) & n > 56. \end{cases}$$

(Recall that the elements of the sequence A(n) are known for $1 \le n \le 56$.) Since $B(n) \ge A(n)$ for any value of $n \in \mathbb{N}$, the growth constant of $B(n)$, if it exists, is an upper bound on λ. Calculations show that $B(n)$ does have an asymptotic growth constant which is slightly smaller than 4.5685, implying the claim. □

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