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Bipartitions of highly connected tournaments

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Abstract

We show that if T is a strongly $10^9k^6\log(2k)$-connected tournament, there exists a partition A, B of $V(T)$ such that each of $T[A]$, $T[B]$ and $T[A, B]$ is strongly k-connected. This provides tournament analogues of two partition conjectures of Thomassen regarding highly connected graphs.

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1 Introduction

1.1 Partitions of highly connected tournaments

The study of graph partitions where the resulting subgraphs inherit the properties of the original graph has a long history with some surprises and numerous open problems, see e.g. the survey [7]. For example, a classical result of Hajnal [1] and Thomassen [9] implies that for every k there exists an integer $f(k)$ such that every $f(k)$-connected graph has a vertex partition into sets A and B so that both A and B induce k-connected graphs. A related conjecture of Thomassen [12] states that for every k there is an $f(k)$ such that every $f(k)$-connected graph G has a bipartition A, B so that the spanning bipartite graph $G[A, B]$ is k-connected. It is not hard to show that one cannot achieve both the above properties simultaneously in a highly connected graph. On the other hand, our main result states that for tournaments T, we can find a single partition which achieves both the above properties. Below we denote by $T[A, B]$ the bipartite subdigraph of T which consists of all edges between A and B but no others.

Theorem 1.1 *Let T be a tournament and $k \in \mathbb{N}$. If T is strongly $10^9 k^6 \log(2k)$-connected, there exists a partition V_1, V_2 of $V(T)$ such that each of $T[V_1]$, $T[V_2]$ and $T[V_1, V_2]$ is strongly k-connected.*

We have made no attempt to optimize the bound on the connectivity in Theorem 1.1. (It would be straightforward to obtain minor improvements at the expense of more careful calculations.) On the other hand, it would be interesting to obtain the correct order of magnitude for the connectivity bound.

Kühn, Osthus and Townsend [4] earlier proved the weaker result that every strongly $10^8 k^6 \log(4k)$-connected tournament T has a vertex partition V_1, V_2 such that $T[V_1]$ and $T[V_2]$ are both strongly k-connected (with some control over the sizes of V_1 and V_2). This proved a conjecture of Thomassen. [4] raised the question whether this can be extended to digraphs.

Our proof of Theorem 1.1 develops ideas in [4]. These in turn are based on the concept of robust linkage structures which were introduced in [2] to prove a conjecture of Thomassen on edge-disjoint Hamilton cycles in highly connected tournaments. Further (asymptotically optimal) results leading on from these approaches were obtained by Pokrovskiy [5,6].

1.2 Subdivisions and linkages

The famous Lovász path removal conjecture states that for every $k \in \mathbb{N}$ there exists $g(k) \in \mathbb{N}$ such that for every pair x, y of vertices in a $g(k)$-connected graph G we can find an induced path P joining x and y in G for which $G \backslash V(P)$ is k-connected. In [11], Thomassen proved a tournament version of this conjecture. We generalize his argument to observe that highly connected tournaments contain a non-separating subdivision of any given digraph H (with prescribed branch vertices). The case when $d = 2$ and $m = 1$ corresponds to the result in [11].

Theorem 1.2 *Let $k, d, m \in \mathbb{N}$. Suppose that T is a strongly $(k + m(d + 2))$-connected tournament, that D is a set of d vertices in T, that H is a digraph on d vertices and m edges and that ϕ is a bijection from $V(H)$ to D. Then T contains a subdivision H^* of H such that*

(i) for each $h \in V(H)$ the branch vertex of H^* corresponding to h is $\phi(h)$,

(ii) $T \setminus V(H^*)$ is strongly k-connected,

(iii) for every edge e of H, the path P_e of H^* corresponding to e is backwards-transitive.

Here a directed path $P = x_1 \dots x_t$ in a tournament T is *backwards-transitive* if $x_i x_j$ is an edge of T whenever $i \geq j + 2$. The graph version of Theorem 1.2 is still open and would follow from the following beautiful conjecture of Thomassen [10].

Conjecture 1.3 *For every $k \in \mathbb{N}$ there exists $f(k) \in \mathbb{N}$ such that if G is a $f(k)$-connected graph and $M \subseteq V(G)$ consists of k vertices then there exists a partition V_1, V_2 of $V(G)$ such that $M \subseteq V_1$, both $G[V_1]$ and $G[V_2]$ are k-connected, and each vertex in V_1 has at least k neighbours in V_2.*

The case $|M| = 2$ would already imply the path removal conjecture. The case $M = \emptyset$ was proved in [3]. It implies the existence of non-separating subdivisions (without prescribed branch vertices) in highly connected graphs. Clearly, Theorem 1.1 implies a tournament version of Conjecture 1.3.

The next theorem guarantees a spanning linkage in a highly connected tournament. It was proved by Thomassen [11] with a super-exponential bound on the connectivity. He asked whether a linear bound suffices. We reduce the bound to a polynomial one. Pokrovskiy [5] showed that a linear bound suffices to guarantee a linkage if we do not require it to be spanning.

Theorem 1.4 *Let $k \in \mathbb{N}$. Suppose that T is a strongly $(k^2 + 3k)$-connected* *tournament and that $x_1, \dots, x_k, y_1, \dots, y_k$ are vertices in T such that $x_i \neq y_i$ for all $i \in [k]$ and all the pairs (x_i, y_i) are distinct. Then T contains pairwise internally disjoint paths P_i from x_i to y_i such that $\{x_1, \dots, x_k, y_1, \dots, y_k\} \cap V(P_i) = \{x_i, y_i\}$ and $V(T) = \bigcup_{i=1}^k V(P_i)$.*

1.3 Sketch of the argument

We now give a brief idea of the argument in the proof of Theorem 1.1 under the much stronger assumptions that $k \gg \log |T|$. In this case we can find $12k$ disjoint sets $A_1, \dots, A_{6k}, B_1, \dots, B_{6k} \subseteq V(T)$ satisfying the following.

1) each A_i and each B_i has size $o(k)$,
2) each A_i induces a transitive subtournament of T with the source a_i,
3) each B_i induces a transitive subtournament of T with the sink b_i,
4) $A_i \setminus \{a_i\}$ is out-dominating and $B_i \setminus \{b_i\}$ is in-dominating.

(Here we say that a set $X \subseteq V(T)$ is *out-dominating* if every vertex $v \in V(T) \setminus X$ has an in-neighbour in X. In-dominating sets are defined similarly.) We now use a result Pokrovskiy [5] which implies that T is $(10^9 k^6 \log(2k)/452)$-linked to find, for each $i \in [6k]$, a path P_i from b_i to a_i such that all the P_i are pairwise disjoint. For each $i \in [6]$ we let $I_i := \{(i-1)k+1, \cdots, ik\}$ and assign vertices to V_1 and V_2 in the following way:

a) $A_i \cup B_i \subseteq V_1$ for $i \in I_1$ and $A_j \cup B_j \subseteq V_2$ for $j \in I_2$,
b) $(A_i \setminus \{a_i\}) \cup (B_j \setminus \{b_j\}) \subseteq V_1$, $a_i, b_j \in V_2$ for all $i \in I_4 \cup I_5$, $j \in I_4 \cup I_6$,
c) $(A_i \setminus \{a_i\}) \cup (B_j \setminus \{b_j\}) \subseteq V_2$, $a_i, b_j \in V_1$ for all $i \in I_3 \cup I_6$, $j \in I_3 \cup I_5$,
d) $V(P_i) \subseteq V_1, V(P_j) \subset V_2$ for all $i \in I_1, j \in I_2$,
e) P_i is a path in $T[V_1, V_2]$ for all $i \in I_3 \cup I_4 \cup I_5 \cup I_6$,
f) we assign the remaining vertices arbitrarily.

(When choosing the paths P_i for $i \in I_3 \cup \ldots \cup I_6$, we will ensure that they have the correct parity in order to guarantee that we can assign the interior vertices on these paths to V_1 and V_2 in such a way that e) holds.) It is now easy to see that each of $T[V_1]$,$T[V_2]$ and $T[V_1, V_2]$ is strongly k-connected. Indeed, consider some $F \subseteq V_1$ with $|F| < k$. So there exists $i \in [k]$ such that F avoids $A_i \cup B_i \cup V(P_i)$. Consider any $x, y \in V_1 \setminus F$. Since B_i is in-dominating, there is an edge from x to some $x' \in B_i$. Similarly, since A_i is out-dominating, there is an edge from some $y' \in A_i$ to y. Then P_i, xx', $y'y$ together with the edge from x' to the sink b_i of B_i and the edge from the source a_i of A_i to y' form a path in $T[V_1 \setminus F]$ from x to y, as required. A similar argument shows that both $T[V_1, V_2], T[V_2]$ are k-connected too.

In general, the problem with this approach is that we cannot guarantee such (small) dominating sets when k is bounded. However, we can still find small sets which dominate a large proportion of $V(T)$. With some new ideas one can use these to ensure strong k-connectivity of both $T[V_1]$, $T[V_2]$ and $T[V_1, V_2]$.

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