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Optimal path and cycle decompositions of dense quasirandom graphs

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Abstract

Motivated by longstanding conjectures regarding decompositions of graphs into paths and cycles, we prove the following optimal decomposition results for random graphs. Let $0 < p < 1$ be constant and let $G \sim G_{n,p}$. Let $odd(G)$ be the number of odd degree vertices in G. Then a.a.s. the following hold:

(i) G can be decomposed into $\lfloor \Delta(G)/2 \rfloor$ cycles and a matching of size $odd(G)/2$.

(ii) G can be decomposed into $\max\{odd(G)/2, \lceil \Delta(G)/2 \rceil\}$ paths.

(iii) G can be decomposed into $\lceil \Delta(G)/2 \rceil$ linear forests.

Each of these bounds is best possible. We actually derive (i)–(iii) from 'quasirandom' versions of our results. In that context, we also determine the edge chromatic number of a given dense quasirandom graph of even order. For all these results, our main tool is a result on Hamilton decompositions of robust expanders by Kühn and Osthus.

Keywords: cycle decomposition, path decomposition, linear arboricity, overfull subgraph conjecture

1 Introduction

There are several longstanding and beautiful conjectures on decompositions of graphs into cycles and/or paths. We consider four of the most well-known in the setting of dense quasirandom and random graphs: the Erdős-Gallai conjecture, the Gallai conjecture on path decompositions, the linear arboricity conjecture as well as the overfull subgraph conjecture.

1.1 Decompositions of random graphs

A classical result of Lovász [18] on decompositions of graphs states that the edges of any graph on n vertices can be decomposed into at most $\lfloor n/2 \rfloor$ cycles and paths. Erdős and Gallai [7,8] made the related conjecture that the edges of every graph G on n vertices can be decomposed into $\mathcal{O}(n)$ cycles and edges. Conlon, Fox and Sudakov [4] recently showed that $\mathcal{O}(n \log \log n)$ cycles and edges suffice and that the conjecture holds for graphs with linear minimum degree. They also proved that the conjecture holds a.a.s. for the binomial random graph $G \sim G_{n,p}$. Korándi, Krivelevich and Sudakov [14] carried out a more systematic study of the problem for $G_{n,p}$: for a large range of p, a.a.s. $G_{n,p}$ can be decomposed into $n/4 + np/2 + o(n)$ cycles and edges, which is asymptotically best possible. They also asked for improved error terms. For constant p, we will give an exact formula.

A further related conjecture of Gallai (see [18]) states that every connected graph on n vertices can be decomposed into $\lceil n/2 \rceil$ paths. The result of Lovász mentioned above implies that for every (not necessarily connected) graph $n-1$ paths suffice. This has been improved to $\lfloor 2n/3 \rfloor$ paths [6,22]. Here we determine the number of paths in an optimal path decomposition of $G_{n,p}$ for constant p. In particular this implies that Gallai's conjecture holds (with room to spare) for almost all graphs.

Next, recall that an edge colouring of a graph is a partition of its edge set into matchings. A matching can be viewed as a forest whose connected components are edges. As a relaxation of this, a *linear forest* is a forest whose components are paths, and the least possible number of linear forests needed to partition the edge set of a graph G is called the *linear arboricity* of G, denoted by $la(G)$. Clearly, in order to cover all edges at any vertex of

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maximum degree, we need at least $\lceil \Delta(G)/2 \rceil$ linear forests. However, for some graphs (e.g. complete graphs on an odd number of vertices) we need at least $\lceil (\Delta(G)+1)/2 \rceil$ linear forests. The following conjecture is known as the linear arboricity conjecture and can be viewed as an analogue to Vizing's theorem.

Conjecture 1.1 (Akiyama, Exoo, Harary [1]) *For every graph G, $la(G) \leq \lceil (\Delta(G)+1)/2 \rceil$.*

This is equivalent to the statement that for all d-regular graphs G, $la(G) = \lceil (d+1)/2 \rceil$. Alon [2] proved an approximate version of the conjecture for sufficiently large values of $\Delta(G)$. Using his approach, McDiarmid and Reed [19] confirmed the conjecture for random regular graphs with fixed degree. We will show that, for a large range of p, a.a.s. the random graph $G \sim G_{n,p}$ can be decomposed into $\lceil \Delta(G)/2 \rceil$ linear forests. Moreover, we use the recent confirmation [5] of the so-called 'Hamilton decomposition conjecture' to deduce that the linear arboricity conjecture holds for large and sufficiently dense regular graphs.

The following theorem summarises our optimal decomposition results for dense random graphs. We denote by $odd(G)$ the number of odd degree vertices in a graph G.

Theorem 1.2 *Let $0 < p < 1$ be constant and let $G \sim G_{n,p}$. Then a.a.s. the following hold:*

(i) G can be decomposed into $\lfloor \Delta(G)/2 \rfloor$ cycles and a matching of size $odd(G)/2$.

(ii) G can be decomposed into $\max\{odd(G)/2, \lceil \Delta(G)/2 \rceil\}$ paths.

(iii) G can be decomposed into $\lceil \Delta(G)/2 \rceil$ linear forests, i.e. $la(G) = \lceil \Delta(G)/2 \rceil$.

Clearly, each of the given bounds is best possible. Moreover, as observed e.g. in [14] for a large range of p, a.a.s. $odd(G_{n,p}) = (1+o(1))n/2$. This means that for fixed $p < 1/2$, the size of an optimal path decomposition of $G_{n,p}$ is determined by the number of odd degree vertices, whereas for $p > 1/2$, the maximum degree is the crucial parameter.

A related result of Gao, Pérez-Giménez and Sato [9] determines the arboricity and spanning tree packing number of $G_{n,p}$. Optimal results on packing Hamilton cycles in $G_{n,p}$ which together cover essentially the whole range of p were proven in [13,15].

One can extend Theorem 1.2(iii) to the range $\frac{\log^{117} n}{n} \leq p = o(1)$ by applying a recent result in [11] on covering $G_{n,p}$ by Hamilton cycles. It would be interesting to obtain corresponding exact results also for (i) and (ii). In particular we believe that the following should hold.

Conjecture 1.3 *Suppose* $p = o(1)$ *and* $\frac{pn}{\log n} \to \infty$. *Then a.a.s.* $G \sim G_{n,p}$ *can be decomposed into* $odd(G)/2$ *paths.*

By tracking the number of cycles in the decomposition constructed in [14] and by splitting every such cycle into two paths, one immediately obtains an approximate version of Conjecture 1.3. Note that this argument does not yield an approximate version of Theorem 1.2(ii) in the case when p is constant.

1.2 Dense quasirandom graphs

As mentioned earlier, we will deduce Theorem 1.2 from quasirandom versions of these results. For this we will consider the following one-sided version of ε-regularity. Let $0 < \varepsilon, p < 1$. A graph G on n vertices is called *lower-*(p, ε)*-regular* if we have $e_G(S, T) \geq (p - \varepsilon)|S||T|$ for all disjoint $S, T \subseteq V(G)$ with $|S|, |T| \geq \varepsilon n$.

The next theorem is a quasirandom version of Theorem 1.2(i). Similarly, we also prove quasirandom versions of parts (ii) and (iii).

Theorem 1.4 *For all* $0 < p < 1$ *there exist* $\varepsilon, \eta > 0$ *such that for sufficiently large* n*, the following holds: Suppose* G *is a lower-*(p, ε)*-regular graph on* n *vertices. Moreover, assume that* $\Delta(G) - \delta(G) \leq \eta n$ *and that* G *is Eulerian. Then* G *can be decomposed into* $\Delta(G)/2$ *cycles.*

This confirms the following conjecture of Hajós (see [18]) for quasirandom graphs (with room to spare): Every Eulerian graph on n vertices has a decomposition into $\lfloor n/2 \rfloor$ cycles. (It is easy to see that this conjecture implies the Erdős-Gallai conjecture.)

We also apply our approach to edge colourings of dense quasirandom graphs. Recall that in general it is NP-complete to decide whether a graph G has chromatic index $\Delta(G)$ or $\Delta(G) + 1$ (see e.g. [12]). We will show that for dense quasirandom graphs of even order this decision problem can be solved in quadratic time without being trivial. For this, call a subgraph H of G *overfull* if $e(H) > \Delta(G)\lfloor |H|/2 \rfloor$. Clearly, if G contains any overfull subgraph, then $\chi'(G) = \Delta(G) + 1$. The following conjecture is known as the overfull subgraph conjecture and dates back to 1986.

Conjecture 1.5 (Chetwynd, Hilton [3]) *A graph* G *on* n *vertices with* $\Delta(G) > n/3$ *satisfies* $\chi'(G) = \Delta(G)$ *if and only if* G *contains no overfull subgraph.*

This conjecture implies the 1-factorization conjecture, that every regular graph of sufficiently high degree and even order can be decomposed into perfect matchings, which was recently proved for large graphs in [5]. (We refer to [21] for a more thorough discussion of the area.) We prove the overfull subgraph conjecture for quasirandom graphs of even order, even if the maximum degree is smaller than stated in the conjecture, as long as it is linear.

Theorem 1.6 *For all $0 < p < 1$ there exist $\varepsilon, \eta > 0$ such that for sufficiently large n, the following holds: Suppose G is a lower-(p, ε)-regular graph on n vertices and n is even. Moreover, assume that $\Delta(G) - \delta(G) \leq \eta n$. Then $\chi'(G) = \Delta(G)$ if and only if G contains no overfull subgraph. Further, there is a polynomial time algorithm which finds an optimal colouring.*

At the first glance, the overfull subgraph criterion seems not very helpful in terms of time complexity, as it involves all subgraphs of G. (On the other hand, Niessen [20] proved that in the case when $\Delta(G) \geq |G|/2$ there is a polynomial time algorithm which finds all overfull subgraphs.) Our proof of Theorem 1.6 will actually yield a simple criterion whether G is class 1 or class 2. Moreover, the proof is constructive, thus using appropriate running time statements for our tools, this yields a polynomial time algorithm which finds an optimal colouring.

The condition of n being even is essential for our proof as we will colour Hamilton cycles with two colours each. It would be interesting to obtain a similar result for graphs of odd order.

Conjecture 1.7 *For every $0 < p < 1$ there exist $\varepsilon, \eta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Whenever G is a lower-(p, ε)-regular graph on $n \geq n_0$ vertices, where n is odd, and $\Delta(G) - \delta(G) \leq \eta n$, then $\chi'(G) = \Delta(G)$ if and only if $\sum_{x \in V(G)} (\Delta(G) - d_G(x)) \geq \Delta(G)$.*

Note that the condition $\sum_{x \in V(G)} (\Delta(G) - d_G(x)) \geq \Delta(G)$ in Conjecture 1.7 is equivalent to the requirement that G itself is not overfull. Also note that the corresponding question for $G_{n,p}$ is easily solved if p does not tend to 0 or 1 too quickly: It is well-known that in this case a.a.s. $G \sim G_{n,p}$ satisfies $\chi'(G) = \Delta(G)$, which follows from the fact that a.a.s. G has a unique vertex of maximum degree.

2 Proof overviews

Note that our main results concern almost regular graphs. So the key step is to partially decompose a given graph (into paths, cycles or appropriate linear forests) such that the remaining graph is regular. We then apply a result on Hamilton decompositions of regular robust expanders by Kühn and Osthus

[16,17].

2.1 Proof sketch of Theorem 1.4

If an Eulerian graph G has a decomposition into $\Delta(G)/2$ cycles, then any vertex of maximum degree must be contained in any cycle of the decomposition. Let Z contain the vertices of maximum degree in G. We want to find a cycle C that contains Z. A cycle on Z would be desirable, yet too much to hope for. However, suppose we are given a set of vertices S (not necessarily disjoint from Z) such that $G[S \cup Z]$ is lower-ε-regular and has linear minimum degree. Then we can find a Hamilton cycle C in $G[S \cup Z]$. Let G' be obtained from G by removing the edges of C. Hence, when going from G to G', the maximum degree decreases by two. Let Z' contain the vertices of maximum degree in G'. Again, we aim at finding a cycle C' that contains Z'. In addition, if $\delta(G') < \delta(G)$, then we want to make sure that C' does not contain any vertex of degree $\delta(G')$. We achieve this as follows. We find another set S' such that $G[S' \cup Z']$ is lower-ε-regular and has linear minimum degree, and critically, S' is disjoint from S. Then we can take C' to be a Hamilton cycle in $G[S' \cup Z']$. In this way we have reduced the maximum degree by 4 and the minimum degree by at most 2 by removing the edges of two cycles. By repeating this 2-step procedure, we will eventually obtain a dense regular graph which can be decomposed into Hamilton cycles.

2.2 Proof sketch of Theorem 1.6

Roughly speaking, instead of inductively removing cycles, we aim to remove paths in order to make our graph regular and then decompose the regular remainder into Hamilton cycles. We can then simply colour each path with two colours and, since our graph has even order, each Hamilton cycle with two colours. We can translate the condition that G does not contain any overfull subgraph into a simple condition on the degree sequence of G. Together with a classic result on multigraphic degree sequences by Hakimi [10], we find an auxiliary multigraph A on $V(G)$ such that $d_A(x) = \Delta(G) - d_G(x)$ for all $x \in V(G)$. If we removed the edges of a Hamilton path from G joining a and b for every edge $ab \in E(A)$, then the leftover would be a regular graph. However, too many iterations would be needed and we could not ensure that the regular remainder is still dense enough to apply the Hamilton decomposition result in [17]. Therefore, we split $E(A)$ into matchings, and for every such matching M we remove a linear forest from G whose leaves are the vertices covered by M. In order to actually find these linear forests, we observe that lower-(p, ε)-

regular graphs contain ‘spanning linkages’ for arbitrary pairs of vertices.

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