Minimum degrees and codegrees of minimal Ramsey 3-uniform hypergraphs

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Abstract

A uniform hypergraph $H$ is called $k$-Ramsey for a hypergraph $F$, if no matter how one colors the edges of $H$ with $k$ colors, there is always a monochromatic copy of $F$. We say that $H$ is minimal $k$-Ramsey for $F$, if $H$ is $k$-Ramsey for $F$ but every proper subhypergraph of $H$ is not. Burr, Erdős and Lovász [S. A. Burr, P. Erdős, and L. Lovász, *On graphs of Ramsey type*, Ars Combinatoria 1 (1976), no. 1, 167–190] studied various parameters of minimal Ramsey graphs. In this paper we initiate the study of minimum degrees and codegrees of minimal Ramsey 3-uniform hypergraphs. We show that the smallest minimum vertex degree over all minimal $k$-Ramsey 3-uniform hypergraphs for $K_t^{(3)}$ is exponential in some polynomial in $k$ and $t$. We also study the smallest possible minimum codegrees over minimal 2-Ramsey 3-uniform hypergraphs.

Keywords: minimal Ramsey hypergraph, minimum degree and codegree
1 Introduction and New Results

A graph $G$ is said to be Ramsey for a graph $F$ if no matter how one colors the edges of $G$ with two colors, say red and blue, there is a monochromatic copy of $F$. A classical result of Ramsey [11] states that for every $F$ there is an integer $n$ such that $K_n$ is Ramsey for $F$. If $G$ is Ramsey for $F$ but every proper subgraph of $G$ is not Ramsey for $F$, then we say that $G$ is minimal Ramsey for $F$. We denote by $\mathcal{M}_k(F)$ the set of minimal graphs $G$ with the property that no matter how one colors the edges of $G$ with $k$ colors, there is a monochromatic copy of $F$ in it, and refer to these as minimal $k$-Ramsey graphs for $F$. There are many challenging open questions concerning the study of various parameters of minimal $k$-Ramsey graphs for various $F$. The most studied ones are the classical (vertex) Ramsey numbers $r_k(F) := \min_{G \in \mathcal{M}_k(F)} v(G)$ and the size Ramsey number $\hat{r}_k(F) := \min_{G \in \mathcal{M}_k(F)} e(G)$, where $v(G)$ is the number of vertices in $G$ and $e(G)$ is its number of edges. To determine the classical Ramsey number $r_2(K_t)$ is a notoriously difficult problem and essentially the best known bounds are $2^{(1+o(1))t/2}$ and $2^{(2+o(1))t}$ due to Spencer [13] and Conlon [4].

Burr, Erdős and Lovász [1] were the first to study other possible parameters of the class $\mathcal{M}_2(K_t)$. In particular they determined the minimum degree $s_2(K_t) := \min_{G \in \mathcal{M}_2(K_t)} \delta(G) = (t-1)^2$ which looks surprising given the exponential bound on the minimum degree of $K_n$ with $n = r_2(K_t)$ (it is not difficult to see that $K_n \in \mathcal{M}_2(K_t)$). Extending their results, Fox, Grinshpun, Liebenau, Person and Szabó [7] studied the minimum degree $s_k(K_t) := \min_{G \in \mathcal{M}_k(K_t)} \delta(G)$ for more colors showing a general bound on $s_k(K_t) \leq 8(t-1)^6k^3$ and proving quasiquadratic bounds in $k$ on $s_k(K_t)$ for fixed $t$. Further results concerning minimal Ramsey graphs were studied in [2,8,9,12,14].

Here we initiate the study of minimal Ramsey 3-uniform hypergraphs and provide first bounds on various notions of minimum degrees for minimal Ramsey hypergraphs. Generally, an $r$-uniform hypergraph $H$ is a tuple $(V, E)$ with vertex set $V$ and $E \subseteq \binom{V}{r}$ being its edge set. Ramsey’s theorem holds for $r$-uniform hypergraphs as well, as shown originally by Ramsey himself [11], and we write $G \rightarrow (F)_k$ if $G$ is $k$-Ramsey for $F$, i.e. if no matter how one colors the edges of the $r$-uniform hypergraph $G$, there is a monochromatic copy of $F$. We denote by $K_t^{(r)}$ the complete $r$-uniform hypergraph with $t$ vertices, i.e.

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$K_t^{(r)} = ([t], [[t]]_r)$, and by the hypergraph Ramsey number $r_k(F)$ the smallest $n$ such that $K_n^{(r)} \rightarrow (F)_k$. While in the graph case the known bounds on $r_2(K_t)$ are only polynomially far apart, already in the case of 3-uniform hypergraphs the bounds on $r_2(K_t^{(3)})$ differ in one exponent: $2^{c_1 t^2} \leq r_2(K_t^{(3)}) \leq 2^{2^{c_2 t}}$ for some absolute positive constants $c_1$ and $c_2$, and a similar situation occurs for higher uniformities. For further information on Ramsey numbers we refer the reader to the standard book on Ramsey theory [10] and for newer results to the survey of Conlon, Fox and Sudakov [5].

Given $\ell \in [r-1]$, we define the degree $\deg(S)$ of an $\ell$-set $S$ in an $r$-uniform hypergraph $H = (V, E)$ as the number of edges that contain $S$ and we define the minimum $\ell$-degree $\delta_\ell(H) := \min_{S \in (V)_\ell} \deg(S)$. For two vertices $u$ and $v$ we simply write $\deg(u,v)$ for the codegree $\deg(\{u,v\})$. Similar to the graph case we extend verbatim the notion of minimal Ramsey graphs to minimal Ramsey $r$-uniform hypergraphs in a natural way. That is, $\mathcal{M}_k(F)$ is the set of all minimal $k$-Ramsey $r$-uniform hypergraphs $H$, i.e. consisting of those with $H \rightarrow (F)_k$ but $H' \not\rightarrow (F)_k$ for all $H' \subsetneq H$. We define

$$s_{k,\ell}(K_t^{(r)}) := \min_{H \in \mathcal{M}_k(K_t^{(r)})} \delta_\ell(H),$$

which extends the introduced graph parameter $s_k(K_t)$. It will be shown actually that $s_{2,2}(K_t^{(3)})$ is zero and thus it makes sense to ask for the second smallest value of the codegrees. This motivates the following parameter $s'_{k,\ell}(K_t^{(r)})$:

$$s'_{k,\ell}(K_t^{(r)}) := \min_{H \in \mathcal{M}_k(K_t^{(r)})} \left( \min \left\{ \deg_H(S) : S \in \left( V(H) \atop \ell \right), \deg_H(S) > 0 \right\} \right).$$

We prove the following results on the minimum degree and codegree of minimal Ramsey 3-uniform hypergraphs for cliques $K_t^{(3)}$.

**Theorem 1.1** The following holds for all $t \geq 4$ and $k \geq 2$

$$\hat{r}_k(K_{t-1}) \leq s_{k,1}(K_t^{(3)}) \leq k^{20k t^4}. \tag{2}$$

The lower bound $\hat{r}_k(K_{t-1})$ is the size-Ramsey number for $K_{t-1}$ and it was shown by Erdős, Faudree, Rousseau and Schelp [6] that $\hat{r}_k(K_t) = \left( \frac{r_k(K_t)}{2} \right)$. Using the lower bound on $r_k(K_t) \geq 2^{\frac{1-o(1)}{4} k^2 t}$ (see e.g. [5]) we obtain $s_{k,1}(K_t^{(3)}) \geq 2^{\frac{1}{2} k t (1-o(1))}$. 

Theorem 1.2 Let \( t \geq 4 \) be an integer. Then,

\[ s_{2,2}(K_t^{(3)}) = 0 \text{ and } s'_{2,2}(K_t^{(3)}) = (t - 2)^2. \]

Notice that with \( s'_{2,2} \) we ask for the smallest positive codegree, while for \( s_{2,2} \) we also allow the codegree to be zero. This in particular means that in any minimal 2-Ramsey hypergraph \( H \) for \( K_t^{(3)} \) we have that a pair of vertices \( u \) and \( v \) are either not contained in a common edge or have codegree at least \((t - 2)^2\). This might look surprising at the first sight since taking \( K_n^{(3)} \) with \( n = r_2(K_t^{(3)}) \) and then deleting all edges that contain two distinguished vertices gives a non-Ramsey hypergraph.

2 Main tools

2.1 BEL-gadgets

We refer in the following to a coloring without a monochromatic copy of \( F \) as an \( F \)-free coloring. Our first tool is a result that asserts existence of non-\( k \)-Ramsey hypergraphs \( \mathcal{H} \) for \( K_t^{(3)} \) that impose certain structure on all \( K_t^{(3)} \)-free colorings of \( E(\mathcal{H}) \).

Theorem 2.1 Let \( k \geq 2 \) and \( t \geq 4 \) be integers. Let \( H \) be a 3-uniform hypergraph with \( H \not\rightarrow (K_t^{(3)})_k \) and let \( c : E(H) \to [k] \) be a \( k \)-coloring which avoids monochromatic copies of \( K_t^{(3)} \). Then, there exists a 3-uniform hypergraph \( \mathcal{H} \) with the following properties:

(i) \( \mathcal{H} \not\rightarrow (K_t^{(3)})_k \),

(ii) \( \mathcal{H} \) contains \( H \) as an induced subhypergraph, i.e. \( \mathcal{H}[V(H)] = H \),

(iii) for every coloring \( \varphi : E(\mathcal{H}) \to [k] \) without a monochromatic copy of \( K_t^{(3)} \), the coloring of \( H \) under \( \varphi \) agrees with the coloring \( c \), up to a permutation of the \( k \) colors,

(iv) if there are two vertices \( a, b \in V(H) \) with \( \deg_H(a, b) = 0 \) then \( \deg_\mathcal{H}(a, b) = 0 \) as well,

(v) if \( |V(H)| \geq 4 \) then for every vertex \( x \in V(\mathcal{H}) \setminus V(H) \) there exists a vertex \( y \in V(H) \) such that \( \deg_\mathcal{H}(x, y) = 0 \).

This theorem is crucial for our constructions of minimal \( k \)-Ramsey hypergraphs and thus for giving upper bounds on \( s_{k,1}(K_t^{(3)}) \), \( s_{k,2}(K_t^{(3)}) \) and \( s'_{k,2}(K_t^{(3)}) \), respectively. For its proof we first show the existence of a 3-uniform
hypergraph \( \mathcal{H} \) and two edges \( f, e \in E(\mathcal{H}) \) with \( |f \cap e| = 2 \) and \( e(\mathcal{H}[e \cup f]) = 2 \) so that \( \mathcal{H} \) is not \( k \)-Ramsey for \( K_t^{(3)} \) with the property that any \( F \)-free \( k \)-coloring of \( E(\mathcal{H}) \) colors the edges \( e \) and \( f \) differently. Putting several copies of these hypergraphs together in an appropriate way we receive a hypergraph promised by Theorem 2.1 (and we refer to such \( \mathcal{H} \) as BEL-gadgets). In the graph case similar gadgets (called positive/negative signal senders) were given first by Burr, Erdős and Lovász [1] in the case of two colors, and later generalized by Burr, Nešetřil and Rödl [2] and by Rödl and Siggers [12].

### 2.2 Random hypergraphs

The random hypergraph \( H^{(3)}(n, p) \) is the probability space of all labeled 3-uniform hypergraphs on the vertex set \([n]\) with each edge appearing with probability \( p \) independently of all other edges. The following lemma is crucial for the upper bound (2) in Theorem 1.1.

**Lemma 2.2** Let \( t \geq 4 \) and \( k \geq 2 \) be integers. There is a 3-uniform hypergraph \( H \) on \( n = k^{10t+4} \) vertices, which can be written as an edge-disjoint union of \( k \) 3-uniform hypergraphs \( H_1, \ldots, H_k \) with the following properties:

(i) for every \( i \in [k] \), \( H_i \) contains no copies of \( K_t^{(3)} \), and

(ii) for any coloring \( c \) of the edges of the complete graph \( K_n \) with \( k \) colors there exists a color \( x \in [k] \) and \( k \) sets \( S_1, \ldots, S_k \) that induce copies of \( K_{t-1} \) in color \( x \) under the coloring \( c \) such that \( H_1[S_1] \cong \ldots \cong H_k[S_k] \cong K_{t-1}^{(3)} \).

The rough idea of the proof of Lemma 2.2 is to take \( k \) random hypergraphs \( H'_1, \ldots, H'_k \sim H^{(3)}(n, p) \), with \( p \) being chosen appropriately. And then to show that, with positive probability, even after deleting those edges which appear in at least two hypergraphs \( H'_i \) or in a copy of \( K_t^{(3)} \) inside some \( H'_i \), we are left with \( k \) edge-disjoint hypergraphs \( H_1, \ldots, H_k \) that satisfy the conditions above.

For the details we refer the reader to the full version of our paper [3].

### 3 Proof of Theorem 1.1

**Lower bound**

Take a minimal \( k \)-Ramsey hypergraph \( \mathcal{H} \) for \( K_t^{(3)} \) together with a vertex \( v \in V(\mathcal{H}) \) such that \( \text{deg}(v) = \delta(\mathcal{H}) = s_{k,1}(K_t^{(3)}) \). We know that there exists a \( K_t^{(3)} \)-free \( k \)-coloring of \( \mathcal{H} \setminus \{v\} \) which cannot be extended to a \( K_t^{(3)} \)-free \( k \)-coloring of \( \mathcal{H} \). But this implies that \( \text{link}_{\mathcal{H}}(v) \rightarrow (K_{t-1})_k \) holds, where
link$_H(v)$ is the link of $v$, i.e., the graph consisting of all edges $e$ such that $e \cup \{v\} \in E(H)$. Therefore: $s_{k,1}(K_t^{(3)}) = \deg(v) \geq \hat{r}_k(K_{t-1})$.

**Upper bound**

The proof of our upper bound on $s_{k,1}(K_t^{(3)})$ makes use of the BEL-gadgets. We fix a 3-uniform hypergraph $H$ as asserted by Lemma 2.2 and a $K_t^{(3)}$-free $k$-coloring $c$ of $E(H)$ which colors each of the subhypergraphs $H_i$ monochromatically with color $i \in [k]$. Applying Theorem 2.1 for this choice of $H$ and $c$, we obtain a new hypergraph $H'$, that contains $H$ as an induced subhypergraph, and we extend it further to a hypergraph $\mathcal{H}$ by adding one new vertex $v$ with the edges $\{v, a, b\}$ for all $\{a, b\} \in \binom{V(H)}{2}$, i.e. the link of $v$ is link$_H(v) := \binom{V(H)}{2}$. So, $\deg_H(v) = \binom{n}{2} < k^{20kt^4}$ holds. Owing to the assertions on $\mathcal{H}$ we have $\mathcal{H}' \not\rightarrow (K_t^{(3)})_k$. On the other hand one can show $\mathcal{H} \rightarrow (K_t^{(3)})_k$, which follows from Property (ii) of Lemma 2.2. Thus, we conclude that there needs to exist a minimal $k$-Ramsey hypergraph $\mathcal{H}''$ of $K_t^{(3)}$ with $\mathcal{H}' \subseteq \mathcal{H}'' \subseteq \mathcal{H}$ and $0 < \deg_{\mathcal{H}''}(v) < k^{20kt^4}$. \hfill \Box

### 4 Proof of Theorem 1.2

**The size of $s_{2,2}'$**

For the proof of $s_{2,2}'(K_t^{(3)}) \geq (t - 2)^2$ we take a minimal 2-Ramsey hypergraph $H$ for $K_t^{(3)}$ together with two vertices $u$ and $v \in V(H)$ such that $\deg_H(u, v) > 0$. We aim to show that $\deg_H(u, v) \geq (t - 2)^2$, and thus, for contradiction, we assume the opposite. We then delete all edges containing both $u$ and $v$ in order to obtain a hypergraph $H'$, which satisfies $H' \not\rightarrow (K_t^{(3)})_2$. That is, we find a red-blue coloring $c$ of $E(H')$ which does not create a monochromatic copy of $K_t^{(3)}$. Now, let $N(u, v) := \{w \in V(H) : \{u, v, w\} \in E(H)\}$, $\deg_H(u, v) = |N(u, v)|$, and fix a longest sequence $B_1, \ldots, B_k$ of vertex disjoint sets of size $t - 2$ in $N(u, v)$, such that both $B_i \cup \{u\}$ and $B_i \cup \{v\}$ span only blue edges under the coloring $c$ in $H'$. By assumption on the codegree $\deg_H(u, v)$, we know that $k < t - 2$. We then extend the coloring $c$ to a coloring of $E(H)$ as follows. For each edge $e = \{u, v, w\} \in E(H)$ with $w \in \bigcup B_i$, we set $c(e) = \text{red}$, while for all other edges $e = \{u, v, w\} \in E(H)$ we set $c(e) = \text{blue}$. It then follows that under this coloring there is no monochromatic copy of $K_t^{(3)}$ in $H$, contradicting $H \rightarrow (K_t^{(3)})_2$.

For the proof of $s_{2,2}'(K_t^{(3)}) \leq (t - 2)^2$ we first provide a hypergraph $H$ as follows. We choose $V(H) := [(t - 2)^2] \cup \{a, b\}$ together with a partition
of $[(t-2)^2]$ into $(t-2)$ equal-sized sets $V_1, \ldots, V_{t-2}$. Moreover, we define $E(H)$ by taking all edges of the clique $K_t^{(3)}$ on the vertex set $\bigcup V_i \cup \{a, b\}$ and then deleting all edges that contain both $a$ and $b$ plus deleting all edges that cross exactly two different $V_i$s and contain neither $a$ nor $b$. For this particular hypergraph, we then define a red-blue-coloring $c$ as follows: the edges contained in $V_i \cup \{a\}$ and in $V_i \cup \{b\}$ for $i \in [t-2]$ are colored blue, while the other edges of $H$ are colored red. By construction of $H$ this coloring does not produce a monochromatic copy of $K_t^{(3)}$.

Now, applying Theorem 2.1 to $H$ and $c$, we obtain a 3-uniform hypergraph $H$ which contains $H$ as an induced subhypergraph such that $H \not\rightarrow K_t^{(3)}$, $\deg_H(a, b) = 0$ and such that any $K_t^{(3)}$-free red-blue coloring $\phi$ of $E(H)$ agrees on $E(H)$ with the coloring $c$ up to permutation of the two colors. Extending this construction by adding to $H$ all $(t-2)^2$ edges $\{a, b, u\}$ where $u \in [(t-2)^2]$, we finally end up in a hypergraph $H'$ for which it is not difficult to see that $H' \rightarrow (K_t^{(3)})_2$. Thus, as $H \not\rightarrow (K_t^{(3)})_2$, there needs to exist a minimal 2-Ramsey hypergraph $H''$ of $K_t^{(3)}$ with $H \subseteq H'' \subseteq H'$ and such that $0 < \deg_{H''}(a, b) \leq (t-2)^2$, i.e., $s_{2,2}(K_t^{(3)}) \leq (t-2)^2$.

**Showing** $s_{2,2}(K_t^{(3)}) = 0$.

Let us consider the previous construction of $H'$ again. As $s'_{2,2}(K_t^{(3)}) = (t-2)^2$ was proven, we know that any minimal 2-Ramsey subhypergraph of $H'$ for $K_t^{(3)}$ has to contain all $(t-2)^2$ edges that contain $a$ and $b$, and in particular, any such minimal hypergraph $H''$ needs to contain all vertices of the induced subhypergraph $H$. However, $H''[V(H)] \not\rightarrow (K_t^{(3)})_2$ holds, as can be seen by considering a red-blue-edge-coloring chosen uniformly at random and showing that the expected number of monochromatic copies of $K_t^{(3)}$ in $H''[V(H)]$ is less than 1.

Thus, any minimal 2-Ramsey subhypergraph $H''$ of $H'$ has to contain at least one further vertex $x \notin V(H)$. Then, since $|V(H)| = (t-2)^2 + 2 \geq 6$, it follows by Property (v) of Theorem 2.1 that there exists a vertex $y \in V(H)$ such that $0 = \deg_{H'}(x, y) \geq \deg_{H''}(x, y)$, i.e., $s_{2,2}(K_t^{(3)}) = 0$. \qed

For the details we refer the reader to the full version of our paper [3].

**References**


